Det Kgl. Danske Videnskabernes Selskab.
Mathematisk-fysiske Meddelelser. **XVII,** 8.

ON THE FIELD THEORY OF NUCLEAR FORCES

BY

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KØBENHAVN
EJNAR MUNKSGAARD
1940

Printed in Denmark
Bianco Lunos Bogtrykkeri A/S
REPRINTED 1948 TUTEIN OG KOCH

INTRODUCTION

As was first pointed out by YUKAWA[1], the fundamental property of the nuclear forces of having a limited range can be very simply accounted for by the introduction of a new kind of field, generated by the nuclear particles, and through the intermediary of which forces are established between these particles. With such a field is associated, according to the principles of quantum theory, a new kind of particles, the mass of which is connected with the range of the forces[2]. The value of this mass turns out to be intermediate between that of the electron and that of the proton, and actually of the same order of magnitude as that of the new kind of charged particles, called *mesons*[3], found in cosmic radiation.

Charged meson fields can only give rise, in first approximation, to forces between protons and neutrons. Since short range forces of the same order of magnitude have been shown by scattering experiments to act between any pair of protons or neutrons, it seems necessary further to assume the existence of neutral meson fields; and there is also some evidence[4] of such a neutral penetrating component in the cosmic radiation. It is clear that the forces originating from a purely neutral meson field would be exactly independent of the proton or neutron character of the nuclear particles; but there is also, as shown by KEMMER[5], a possibility of combining in a symmetrical way charged and neutral meson fields so as to secure this charge-independence property of the nuclear forces. This last possibility is especially important since it permits to maintain a relation between a field theory of charge-independent nuclear forces and the various effects brought in connection with the occurrence of charged mesons, viz. the relation between the range of the forces and the mass of the charged mesons observed in cosmic rays, the relation between the life-time of these mesons and the decay-constants of β-radioactive nuclei[6)], and the anomalous magnetic moments of the proton and the neutron[7)].

The simplest wave equations for the mesons which satisfy, besides the claim of relativistic invariance, the condition of giving a positive definite expression for the energy, reduce to four types, characterized by different covariance properties of the wave-functions, and each allowing the existence of neutral as well as positively and negatively charged mesons[8)]. The expressions for the nuclear forces resulting from each of these types of meson fields have hitherto been discussed by using the ordinary perturbation method of quantum theory and taking into consideration only the first non-vanishing approximation, in spite of the well-known lack of convergence of the method. Our first task will be to examine more closely the reliability of such results, and for this purpose we shall use a method of canonical transformation[9)] quite analogous to that used in electrodynamics to separate from the expression of the total energy of a system consisting of electrons and an electromagnetic field, a term depending only on the coordinates of the electrons and representing the Coulomb potential energy. For a system consisting of nuclear particles and any meson field, it is, in fact, possible, as we shall show in the first part of the present paper, to find canonical transformations effecting the separation of a "static" interaction between the nuclear particles, defined as the part of the interaction which is obtained when one neglects the time-variations of the variables characterizing the positions, spins and isotopic spins of the heavy particles. The expression for this static interaction is found to be in all cases just the same as that obtained as a first approximation in the perturbation method and contains three kinds of static potentials, viz. besides a spin-independent potential and a spin-spin coupling such as have hitherto been mainly used in the description of nuclear forces, a further directional coupling of the type of a dipole interaction.

The next question, which will be discussed in the second part of this paper, is that of the fixation of the choice hitherto left open between the four possible types of meson fields, and of the possibility of sharply delimiting a region in which the formalism thus arrived at, which has of course all the defects inherent in any quantum field theory, can be applied unambiguously. Above all, it must be observed that the static potential of dipole interaction type is so strongly singular for infinitely small mutual distances of the nuclear particles that it would not in general allow the existence of stationary states for a system of such particles. In view of the provisory character of the whole theory, it might be attempted to remove this difficulty by taking recourse to some "cutting-off" prescription, consisting, for example, in replacing the interaction energy of a pair of nuclear particles by some constant potential for all values of the mutual distance of the particles smaller than a conveniently chosen value[10]. Quite apart from the arbitrariness involved in the fixation by means of some special properties of nuclear systems of a cutting-off radius which should be connected with the general difficulties of the quantization of fields, it must be stressed that in any such theory non-static effects which do not arise from the field quantization would occur to an extent sufficient to make the exclusive use of the static forces illusory in the determination of stationary states of nuclei. As will be shown with more detail in the second part of this paper, these effects are due to the time-variations of the spins and isotopic spins of the interacting particles which, in first approximation, take the simple form of precessions; the quantitative treatment of the corresponding contributions to the total energy is made impossible by the circumstance that they cannot be unambiguously separated from the infinite terms always present in a quantum field theory. It must therefore be concluded that a satisfactory field theory of nuclear forces must be such as not to give rise to any static potential of the dipole interaction type.

A further requirement restricting the choice of the type of meson field to be adopted is the condition that the interaction between a proton and a neutron should lead to the correct positions of the ^3S ground level and excited ^1S-level of the deuteron, known from experiment. These two conditions cannot be satisfied with one type of meson field only, but it will be seen that, if we take Kemmer's symmetrical combination of charged and neutral fields, there is a definite mixture of two types of meson fields, viz. a vector meson field and a pseudoscalar meson field, for which the resulting static interactions are compatible with the requirements of the empirical deuteron spectrum without containing any singular terms, and in which the precession effects just described become therefore negligible. In such a theory, it is also possible to apply in a consistent way to the Hamiltonian obtained after performing the separation of the static interactions by means of the canonical transformation mentioned above, a prescription regarding the interpretation of the formalism, analogous to the so-called correspondence treatment of quantum electrodynamics[11]. Needless to say, this prescription must include the essential restriction, pointed out by HEISENBERG[12], of the scope of the formalism to processes involving only energies not large compared with the rest-energy of the mesons.

In the third part of the present paper, we apply the theory just outlined to the discussion of the stationary states of the deuteron, including the calculation of the electric quadrupole moment of the ground state. As regards this last property, its experimental discovery by RABI and his collaborators[13] is of considerable theoretical importance, since it clearly shows that the forces acting between a proton and a neutron must to a quite appreciable extent depend on the spatial orientations of the spins of the heavy particles. It is therefore a satisfactory feature of the present form of the meson theory that it actually provides such a directional coupling, arising from non-static interaction terms, which permits a complete treatment of the problem[14].

Finally, we should like briefly to mention the bearing of the above considerations on the theory of β-disintegration obtained by introducing, as proposed by YUKAWA[1], an additional interaction between the meson fields and electrons and neutrinos. In the first place, our transformed Hamiltonian will contain terms which represent a direct interaction between heavy and light particles, and which, when treated as a small perturbation, immediately give the probabilities of β-disintegration processes. It may be regarded as a satisfactory feature of our point of view that, contrary to previous treatments, where the nuclear forces came out in the same stage of the perturbation method as the probabilities of β-decay, an exact account can here be taken at the outset of the main part of these forces to determine the stationary states of the nuclei involved in the β-decay processes. It can further be seen[14a)] that the present theory, involving a mixture of two independent meson fields, provides a possibility of avoiding the serious difficulty pointed out by Nordheim[15)] which affects any theory using only one type of meson field and which consists in a quantitative discrepancy between the observed and the theoretical value of the ratio of the life-time of free mesons to that of light β-radioactive elements. A detailed discussion of the problems of β-disintegration will be published later, in collaboration with S. Rozental.

PART I.

Static nuclear forces.

In the first part of the present paper, we shall be concerned with the determination of the static part of the nuclear forces due to any one of the four types of meson fields shown by KEMMER[8] to satisfy, besides the claim of relativistic invariance, the condition of giving the eigenvalues 0 or 1 for the spin, and a positive definite expression for the energy of the mesons. In each case, we have, as explained in the Introduction, to consider both charged and neutral meson fields. The attribution of an electric charge to the mesons demands the use of complex wave functions. In fact, only with the help of such complex wave functions is it possible to construct an expression for the charge and current density satisfying the continuity equation; and this expression then leads automatically to the existence of both positively and negatively charged mesons. On the other hand, neutral mesons can simply be described by real wave functions[16]. We have thus on the whole to consider in each case three non-interfering meson fields, corresponding to charged and neutral mesons, and represented by three independent sets of real wave functions of the appropriate covariance character.

Let us denote any three such sets of real field quantities by F_1, F_2, F_3 (a whole set of tensor components will for the moment be denoted by a single letter), the index **3** referring to the neutral field, while the indices **1** and **2** refer to the two other real fields which together describe the charged mesons. We may conveniently group corresponding components of these sets into symbolical vectors denoted by

$$\boldsymbol{F} = (F_1, F_2, F_3),$$

and this notation may be extended to the densities

$$\boldsymbol{S} = (S_1, S_2, S_3)$$

of the source distributions giving rise to the real fields in question. Any source density can further be expressed as a sum of the contributions from the different nuclear particles:

$$\boldsymbol{S} = \sum_i \boldsymbol{S}^{(i)},$$

where $\boldsymbol{S}^{(i)}$ denotes the contribution of the i-th nuclear particle.

As shown by Kemmer[5)], the combination of charged and neutral meson fields can be chosen in such a way as to secure that the resulting nuclear forces be completely independent of the proton or neutron character of the particles in all states of the system which are antisymmetric with respect to space and spin coordinates. This is simply effected by taking for any contribution $\boldsymbol{S}^{(i)}$ of a nuclear particle to a source density $\boldsymbol{S}$ an expression of the form

$$\boldsymbol{S}^{(i)} = \boldsymbol{\tau}^{(i)} \cdot s^{(i)},$$

i. e. the product of some operator $s^{(i)}$, which is the same for the three real fields, by the isotopic spin vector

$$\boldsymbol{\tau}^{(i)} = \left(\tau_1^{(i)},\ \tau_2^{(i)},\ \tau_3^{(i)}\right)$$

of the nuclear particle, chosen in such a way that the eigenvalue $+1$ of $\tau_3^{(i)}$ refer to the neutron states of the particle and the eigenvalue -1 to its proton states; the choice of $\tau_3^{(i)}$ for this purpose being, of course, necessarily connected with our attribution of the index **3** to the neutral meson field. For the reasons stated in the Introduction, we shall adopt this symmetrical form of the theory in the following treatment.

We shall begin with the case of the vector meson field, which has hitherto been most extensively studied[17] and which, on account of its similarity with the electromagnetic field, is perhaps more suited for the exposition of the method of derivation of the static nuclear forces.

1. Survey of the formalism of the vector meson theory.

For the description of each of the three real vector meson fields, we have to introduce a four-vector* $(\vec{U}, V)$ and an antisymmetric tensor $(\vec{F}, \vec{G})$; the charged mesons will thus be described by two independent sets of such vectors and tensors:

$$\left(\vec{U}_1,\ V_1\right),\ \left(\vec{F}_1,\ \vec{G}_1\right);\quad \left(\vec{U}_2,\ V_2\right),\ \left(\vec{F}_2,\ \vec{G}_2\right)$$

and the neutral field by a third set

$$\left(\vec{U}_3,\ V_3\right),\quad \left(\vec{F}_3,\ \vec{G}_3\right).$$

With the notation introduced above, all field components can be compactly expressed as

* The arrow indicates a vector in ordinary space.

$$\left(\vec{\boldsymbol{U}},\ \boldsymbol{V}\right),\quad \left(\vec{\boldsymbol{F}},\ \vec{\boldsymbol{G}}\right),$$

and they satisfy the following system of equations*:

$$\left\{\begin{aligned} \dot{\vec{\boldsymbol{U}}} &= -\vec{\boldsymbol{F}} - \operatorname{grad} \boldsymbol{V} + \vec{\boldsymbol{T}} \\ \dot{\vec{\boldsymbol{F}}} &= \kappa^2 \vec{\boldsymbol{U}} + \operatorname{rot} \vec{\boldsymbol{G}} - \vec{\boldsymbol{M}}, \end{aligned}\right\} \tag{1}$$

$$\left\{\begin{aligned} \kappa^2 \boldsymbol{V} &= -\operatorname{div} \vec{\boldsymbol{F}} + \boldsymbol{N} \\ \vec{\boldsymbol{G}} &= \operatorname{rot} \vec{\boldsymbol{U}} + \vec{\boldsymbol{S}}, \end{aligned}\right\} \tag{2}$$

where $\frac{1}{\kappa}$ denotes the range of the nuclear forces. The four-vectors $(\vec{\boldsymbol{M}}, \boldsymbol{N})$ and the antisymmetric tensors $(\vec{\boldsymbol{T}}, \vec{\boldsymbol{S}})$ represent the densities of the source distributions of the meson fields according to the following definitions, which refer to the description of the state of the heavy particles in their configuration space $(\vec{x}^{(1)}, \vec{x}^{(2)}, \cdots, \vec{x}^{(i)} \cdots)$:

$$\left\{\begin{aligned} \vec{\boldsymbol{M}} &= \sum_i \vec{\boldsymbol{M}}^{(i)} = g_1 \sum_i \tau^{(i)} \vec{\alpha}^{(i)} \delta\left(\vec{x} - \vec{x}^{(i)}\right) \\ \boldsymbol{N} &= \sum_i \boldsymbol{N}^{(i)} = g_1 \sum_i \tau^{(i)} \qquad \delta\left(\vec{x} - \vec{x}^{(i)}\right), \end{aligned}\right\} \tag{3}$$

$$\left\{\begin{aligned} \vec{\boldsymbol{T}} &= \sum_i \vec{\boldsymbol{T}}^{(i)} = -\frac{g_2}{\kappa} \sum_i \tau^{(i)} \rho_2^{(i)} \vec{\sigma}^{(i)} \delta\left(\vec{x} - \vec{x}^{(i)}\right) \\ \vec{\boldsymbol{S}} &= \sum_i \vec{\boldsymbol{S}}^{(i)} = \frac{g_2}{\kappa} \sum_i \tau^{(i)} \rho_3^{(i)} \vec{\sigma}^{(i)} \delta\left(\vec{x} - \vec{x}^{(i)}\right); \end{aligned}\right\} \tag{4}$$

the matrices $\rho^{(i)}, \vec{\sigma}^{(i)}$ and $\vec{\alpha}^{(i)} = \rho_1^{(i)} \vec{\sigma}^{(i)}$ are the usual Dirac matrices belonging to the i-th nuclear particle, while the constants g_1, g_2, which have both the dimension of an electric charge, determine the strength of the sources of

* The notation $\dot{A}$ represents the time derivative of A, divided by the velocity of light.

the meson fields and consequently the magnitude of the nuclear forces. It must be noticed that, in contrast to $\boldsymbol{N}$ and $\vec{\boldsymbol{S}}$, the components $\vec{\boldsymbol{M}}$ and $\vec{\boldsymbol{T}}$ contain a factor of the same order of magnitude as the ratio between the velocities of the nuclear particles and the velocity of light (which we will express by saying that they are "of the first order in the velocities").

The field equations (1), as well as the equations of motion for the heavy particles, which we need not write down explicitely, may be derived as canonical equations from a Hamiltonian

$$\mathcal{H} = \mathcal{H}_k + \mathcal{H}_F, \tag{5}$$

where

$$\mathcal{H}_k = \sum_i \left\{ \vec{\alpha}^{(i)} \vec{p}^{(i)} + \rho_3^{(i)} \left(\frac{1+\tau_3^{(i)}}{2} M_N c^2 + \frac{1-\tau_3^{(i)}}{2} M_P c^2 \right) \right\} \tag{6}$$

is the kinetic energy of the nuclear particles, and*

$$\left.\begin{aligned} \mathcal{H}_F = \frac{1}{2} \int \left\{ \vec{\boldsymbol{F}}^2 + \kappa^2 \boldsymbol{V}^2 + \vec{\boldsymbol{G}}^2 + \kappa^2 \vec{\boldsymbol{U}}^2 \right\} dv \\ - \int \left\{ \vec{\boldsymbol{M}} \cdot \vec{\boldsymbol{U}} + \vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{F}} \right\} dv \end{aligned}\right\} \tag{7}$$

the meson field energy, including the interaction with the nuclear particles; in the expression (6), M_N and M_P denote the masses of the neutron and the proton respectively, and $\vec{p}^{(i)}$ represents the momentum of the i-th particle multiplied by the velocity of light. The canonical variables $\left(\vec{p}^{(i)}, \vec{x}^{(i)}\right)$ of the nuclear particles and $\left(-\vec{\boldsymbol{F}}, \vec{\boldsymbol{U}}\right)$ of the meson fields satisfy the commutation rules

* The notation $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}}$ (and likewise $\vec{\boldsymbol{A}}^2$) represents a double scalar product, i. e. a double summation $\sum_\mu \sum_{\boldsymbol{m}} A^\mu_{\boldsymbol{m}} B^\mu_{\boldsymbol{m}}$ over the ordinary space indices μ and the symbolic space indices $\boldsymbol{m}$.

$$\left.\begin{aligned} \left[p^{(i)\mu},\ x^{(k)\nu}\right] &= \frac{\hbar c}{i}\,\delta^{(ik)}\,\delta^{\mu\nu} \\ \left[U^{\mu}_{m}\left(\vec{x},\,t\right),\ F^{\nu}_{n}\left(\vec{x}',t\right)\right] &= \frac{\hbar c}{i}\,\delta\left(\vec{x}-\vec{x}'\right)\delta^{\mu\nu}\,\delta_{mn}, \end{aligned}\right\} \tag{8}$$

all other pairs commuting; their time-derivatives are then calculated by the usual rule

$$\dot{A} = \frac{i}{\hbar c}\left[\mathscr{H},\ A\right],$$

$\boldsymbol{V}$ and $\vec{\boldsymbol{G}}$ being regarded as functions of the dynamical variables defined by the equations (2). Although this Hamiltonian scheme appears very unsymmetrical, it can be verified that it satisfies the requirement of relativistic invariance.

2. Separation of the static nuclear forces in the vector meson theory.

The analogy of the equations (1) and (2) with the Maxwell equations of an electromagnetic field suggests in the first place to consider as the static parts of the meson fields the solutions of the equations

$$\left\{\begin{aligned} \vec{\boldsymbol{F}}^{\circ} &= -\operatorname{grad}\boldsymbol{V}^{\circ} \\ \operatorname{div}\vec{\boldsymbol{F}}^{\circ} + \kappa^2\boldsymbol{V}^{\circ} &= \boldsymbol{N} \end{aligned}\right\} \tag{9}$$

$$\left\{\begin{aligned} \vec{\boldsymbol{G}}^{\circ} &= \operatorname{rot}\vec{\boldsymbol{U}}^{\circ} + \vec{\boldsymbol{S}} \\ \operatorname{rot}\vec{\boldsymbol{G}}^{\circ} + \kappa^2\vec{\boldsymbol{U}}^{\circ} &= 0 \end{aligned}\right\} \tag{10}$$

obtained from (1) and (2) by cancelling the time-derivatives $\dot{\vec{\boldsymbol{U}}}$, $\dot{\vec{\boldsymbol{F}}}$ and the quantities $\vec{\boldsymbol{T}}$ and $\vec{\boldsymbol{M}}$, which are proportional to the velocities of the nuclear particles. Strictly speaking, we should also in the expression (4) for $\vec{\boldsymbol{S}}$ replace the $\rho_3^{(i)}$

by 1; but, in order to obtain a more symmetrical treatment, we prefer to retain the small differences $\rho_3^{(i)} - 1$, which are only of the second order in the velocities. The solution of (9) and (10) is readily reduced—by using the condition $\operatorname{div} \vec{\boldsymbol{U}}^\circ = 0$ which follows from the second equation (10)—to that of the equations

$$\left\{ \begin{array}{l} \Delta \boldsymbol{V}^\circ - \kappa^2 \boldsymbol{V}^\circ = -\boldsymbol{N} \\ \Delta \vec{\boldsymbol{U}}^\circ - \kappa^2 \vec{\boldsymbol{U}}^\circ = \operatorname{rot} \vec{\boldsymbol{S}}. \end{array} \right\} \tag{11}$$

With the help of the Green function

$$\varphi(r) = \frac{1}{4\pi} \frac{e^{-\kappa r}}{r}, \quad \left(r = \left| \vec{x} - \vec{x}' \right| \right) \tag{12}$$

which satisfies the equation

$$\Delta \varphi - \kappa^2 \varphi = -\delta \left(\vec{x} - \vec{x}' \right), \tag{13}$$

we obtain immediately

$$\left\{ \begin{array}{l} \boldsymbol{V}^\circ \left(\vec{x} \right) = \int \boldsymbol{N} \left(\vec{x}' \right) \varphi(r)\, dv' \\ \vec{\boldsymbol{U}}^\circ \left(\vec{x} \right) = -\int \operatorname{rot} \vec{\boldsymbol{S}} \left(\vec{x}' \right) \varphi(r)\, dv', \end{array} \right\} \tag{14}$$

from which we derive $\vec{\boldsymbol{F}}^\circ$ and $\vec{\boldsymbol{G}}^\circ$ by means of the first equations (9) and (10).

We may now define new field variables $\vec{\boldsymbol{U}}^1$, $\vec{\boldsymbol{F}}^1$ by putting

$$\vec{\boldsymbol{U}} = \vec{\boldsymbol{U}}^\circ + \vec{\boldsymbol{U}}^1, \quad \vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}^\circ + \vec{\boldsymbol{F}}^1. \tag{15}$$

If we insert these expressions into the field Hamiltonian $\mathscr{H}_F$, we find that it separates exactly in the form

$$\mathscr{H}_F = \mathscr{H}_F^\circ + \mathscr{H}_F^1 + \mathscr{W}_F, \tag{16}$$

where the first term,

$$\mathcal{H}_F^\circ = \frac{1}{2}\int\left\{\left(\vec{\boldsymbol{F}}^\circ\right)^2 + \kappa^2\left(\boldsymbol{V}^\circ\right)^2\right\} dv + \frac{1}{2}\int\left\{\left(\vec{\boldsymbol{G}}^\circ\right)^2 + \kappa^2\left(\vec{\boldsymbol{U}}^\circ\right)^2\right\} dv\,, \quad (17)$$

is a function of the coordinates of the heavy particles alone, while the second,

$$\mathcal{H}_F^1 = \frac{1}{2}\int\left\{\left(\vec{\boldsymbol{F}}^1\right)^2 + \kappa^{-2}\left(\operatorname{div}\vec{\boldsymbol{F}}^1\right)^2 + \left(\operatorname{rot}\vec{\boldsymbol{U}}^1\right)^2 + \kappa^2\left(\vec{\boldsymbol{U}}^1\right)^2\right\} dv\,, \quad (18)$$

has the same form as the Hamiltonian of a meson field in free space; the last term,

$$\mathcal{W}_F = -\int\left\{\vec{\boldsymbol{M}}\left(\vec{\boldsymbol{U}}^\circ + \vec{\boldsymbol{U}}^1\right) + \vec{\boldsymbol{T}}\left(\vec{\boldsymbol{F}}^\circ + \vec{\boldsymbol{F}}^1\right)\right\} dv, \quad (19)$$

which represents both a direct interaction between the heavy particles and an interaction between these particles and the non-static meson fields, is only of the first order in the velocities. In fact, the remaining cross-terms, which occur when the substitution (15) is carried out,

$$\int\left\{\vec{\boldsymbol{F}}^\circ\,\vec{\boldsymbol{F}}^1 - \boldsymbol{V}^\circ\operatorname{div}\vec{\boldsymbol{F}}^1\right\} dv + \int\left\{\vec{\boldsymbol{G}}^\circ\operatorname{rot}\vec{\boldsymbol{U}}^1 + \kappa^2\,\vec{\boldsymbol{U}}^\circ\,\vec{\boldsymbol{U}}^1\right\} dv,$$

reduce by partial integrations to

$$\int\left\{\vec{\boldsymbol{F}}^\circ + \operatorname{grad}\boldsymbol{V}^\circ\right\}\vec{\boldsymbol{F}}^1\, dv + \int\left\{\operatorname{rot}\vec{\boldsymbol{G}}^\circ + \kappa^2\,\vec{\boldsymbol{U}}^\circ\right\}\vec{\boldsymbol{U}}^1\, dv,$$

i. e. to zero in virtue of the equations (9) and (10).

Now, it must be observed that, according to the definitions (15), the different components of $\vec{\boldsymbol{U}}^1$, $\vec{\boldsymbol{F}}^1$ do not commute with all the coordinates of the heavy particles, so that the terms of the Hamiltonian relating to the nuclear particles alone are not commutable with $\mathcal{H}_F^1$ and that, therefore, not the whole interaction of the system is represented by the terms $\mathscr{H}_F^\circ$ and $\mathscr{W}_F$. This incompleteness of separation arises, however, only from the non-commutability of the matrices $\vec{\sigma}^{(i)}$, $\rho^{(i)}$, $\tau^{(i)}$ and, if one could look apart from this non-commutability, the variables $\vec{U}^1$, $\vec{F}^1$ would even satisfy the canonical commutation rules. The separation (15) would then be part of a canonical transformation which, applied to $\mathscr{H}_F$, would also effect a separation of this function into terms of direct interaction between the heavy particles and the Hamiltonian (18) of pure meson fields, with a small remaining interaction between these fields and the heavy particles.

Let $\mathscr{S}$ denote the unitary operator of such a canonical transformation, defining any new variable A' in terms of the old variables by the formula $A' = \mathscr{S}^{-1} A\, \mathscr{S}$. The Hamiltonian of the system in terms of the new variables is then simply given by the expression $\mathscr{S}'\, \mathscr{H}'\, \mathscr{S}'^{-1}$, where $\mathscr{S}'$ and $\mathscr{H}'$ are the same functions of the new variables as the functions $\mathscr{S}$ and $\mathscr{H}$ of the old variables, the latter function being defined by (5), (6), (7); we have, of course, identically $\mathscr{S}' = \mathscr{S}$. The neglect of the non-commutability of the matrices $\vec{\sigma}^{(i)'}$, $\rho^{(i)'}$, $\tau^{(i)'}$ in the calculation of such an expression means neglecting some terms which contain linearly the commutators of these matrices with $\mathscr{H}'$, i. e. on account of the relation

$$[A', \mathscr{H}'] = \mathscr{S}^{-1}\, [A, \mathscr{H}]\, \mathscr{S} = \hbar c i\, \mathscr{S}^{-1} \dot{A}\, \mathscr{S}$$

the time-variations of the matrices $\vec{\sigma}^{(i)}$, $\rho^{(i)}$, $\tau^{(i)}$. If we therefore conveniently define as static interactions those which are independent of the time-variations $\dot{\vec{x}}^{(i)} = \vec{\alpha}^{(i)}$, $\dot{\vec{\sigma}}^{(i)}$, $\dot{\rho}^{(i)}$, $\dot{\tau}^{(i)}$ of the variables of the nuclear particles*,

* In his papers cited above[9]), E. Stückelberg proposes a definition of the expression "static interaction" which, as he also points out in a

we see that the canonical transformation considered will lead to the separation of all such static interaction terms in the Hamiltonian. The direct interaction term contained in $\mathcal{W}_F$ being obviously of non-static character, the energy of static interaction of the heavy particles is thus simply obtained by putting in the expression (17) for $\mathcal{H}_F^\circ$ all $\rho_3^{(i)}$ equal to 1; the remaining non-static part of $\mathcal{H}_F^\circ$ is only of the second order in the velocities.

The explicit expression of the operator $\mathcal{S}$ of a canonical transformation which, under neglect of the non-commutability of $\vec{\sigma}^{(i)}$, $\rho^{(i)}$, $\tau^{(i)}$, contains the formulae (15), is easily verified to be*

$$\left.\begin{aligned} \mathcal{S} &= e^{\frac{i}{\hbar c}\mathcal{K}_F}, \\ \text{with} \qquad \mathcal{K}_F &= \int\left\{\vec{F}^\circ\,\vec{U} - \vec{U}^\circ\,\vec{F}\right\} dv. \end{aligned}\right\} \tag{20}$$

We have in fact

$$\vec{U}' = \mathcal{S}^{-1}\,\vec{U}\,\mathcal{S} = \vec{U} + \mathcal{S}^{-1}\left[\vec{U}, \mathcal{S}\right] = \vec{U} + \frac{i}{\hbar c}\left[\vec{U}, \mathcal{K}_F\right]$$
$$= \vec{U} - \vec{U}^\circ = \vec{U}^1,$$

and similarly

$$\vec{F}' = \vec{F} - \vec{F}^\circ = \vec{F}^1,$$

since $\left[\vec{A}, \mathcal{S}\right] = \mathcal{S}\,\frac{i}{\hbar c}\left[\vec{A}, \mathcal{K}_F\right]$, when $\left[\vec{A}, \mathcal{K}_F\right]$ is commutable with $\mathcal{K}_F$, which is the case for $\vec{A} = \vec{U}$ and $\vec{A} = \vec{F}$,

note to "Nature", **143**, 560 (1939), differs from the definition adopted here by excluding only the terms depending on the $\dot{\vec{x}}$. As will appear in Part II, the present definition would seem more convenient for the formulation of the restrictions to be imposed on the physical interpretation of the formalism.

* Of course, the operator $\mathcal{S}$ is not uniquely determined; another possible choice would be, for example,

$$\mathcal{S} = e^{\frac{i}{\hbar c}\int \vec{F}^\circ\,\vec{U}\,dv}\cdot e^{-\frac{i}{\hbar c}\int \vec{U}^\circ\,\vec{F}\,dv}.$$

About this point, see p. 37.

when we neglect the non-commutability of spin and isotopic spin operators. Always with this last restriction, the transformation $\mathcal{S}$ leaves all the variables of the nuclear particles unchanged with the exception of the momenta $\vec{p}^{(i)}$. In terms of the new variables, the Hamiltonian of the system is given by

$$\mathcal{S}' \mathcal{H}' \mathcal{S}'^{-1} = \mathcal{S}' \mathcal{H}_k' \mathcal{S}'^{-1} + \mathcal{S}' \mathcal{H}_F' \mathcal{S}'^{-1},$$

where $\mathcal{S}'$, as already stated, is the same function of the new variables as the function $\mathcal{S}$ of the old variables defined by (20). The term $\mathcal{S}' \mathcal{H}_k' \mathcal{S}'^{-1}$ differs from the kinetic energy $\mathcal{H}_k'$ ~~of the~~ heavy particles only through new interaction terms

$$\mathcal{S}'^{-1} \left[\mathcal{H}_k', \mathcal{S}' \right] = \sum_i \vec{\alpha}^{(i)} \cdot \mathcal{S}'^{-1} \left[\vec{p}^{(i)'}, \mathcal{S}' \right]$$

of the first order in the velocities, while of course

$$\mathcal{S}' \mathcal{H}_F' \mathcal{S}'^{-1} = \mathcal{H}_F^{\circ} + \mathcal{H}_F^{1} + \mathcal{W}_F$$

with the definitions (17), (18) and (19).

3. Pseudoscalar meson theory.

The method explained in the preceding section may immediately be applied to any other type of meson field. Since the procedure is entirely similar in all cases, we shall in this section give a brief treatment only of the pseudoscalar meson theory, which, as stated in the Introduction, will be used extensively in the following. The field components here consist of pseudoscalars Ψ and pseudo-four-vectors $(\vec{\Gamma}, \Phi)$ satisfying the equations

$$\left\{\begin{array}{rl} \dot{\boldsymbol{\Psi}} &= \boldsymbol{\Phi} - \boldsymbol{Q} \\ -\dot{\boldsymbol{\Phi}} &= \kappa^2 \boldsymbol{\Psi} + \operatorname{div} \vec{\boldsymbol{\Gamma}} - \boldsymbol{R}, \end{array}\right\} \tag{21}$$

$$\vec{\boldsymbol{\Gamma}} = -\operatorname{grad} \boldsymbol{\Psi} + \vec{\boldsymbol{P}}, \tag{22}$$

in which the sources of the field are represented by the density functions

$$\boldsymbol{R} = f_1 \sum_i \overset{(i)}{\boldsymbol{\tau}} \overset{(i)}{\rho_2} \delta\left(\vec{x} - \vec{x}^{(i)}\right), \tag{23}$$

$$\left\{\begin{array}{l} \vec{\boldsymbol{P}} = \frac{f_2}{\kappa} \sum_i \overset{(i)}{\boldsymbol{\tau}} \overset{(i)}{\vec{\sigma}} \delta\left(\vec{x} - \vec{x}^{(i)}\right), \\ \boldsymbol{Q} = \frac{f_2}{\kappa} \sum_i \overset{(i)}{\boldsymbol{\tau}} \overset{(i)}{\rho_1} \delta\left(\vec{x} - \vec{x}^{(i)}\right), \end{array}\right\} \tag{24}$$

transforming respectively as pseudoscalars and pseudo-four-vectors; the constants f_1 and f_2 are again chosen so as to have the dimensions of an electric charge. The quantities $\boldsymbol{R}$ and $\boldsymbol{Q}$ are of the first order in the velocities.

The field equations (21), with $\vec{\boldsymbol{\Gamma}}$ defined by (22), appear as canonical equations if we regard the $\boldsymbol{\Psi}$'s as canonical variables with conjugate momenta $\boldsymbol{\Phi}$, obeying the commutation rules

$$\left[\Phi_m\left(\vec{x}, t\right), \ \Psi_n\left(\vec{x}', t\right)\right] = \frac{\hbar c}{i} \delta\left(\vec{x} - \vec{x}'\right) \cdot \delta_{mn}, \tag{25}$$

etc., and if we take for the field Hamiltonian, including interaction with the nuclear particles,

$$\left.\begin{array}{l} \mathcal{H}_\Phi = \frac{1}{2}\int\left\{\boldsymbol{\Phi}^2 + \left(\vec{\boldsymbol{\Gamma}}\right)^2 + \kappa^2 \boldsymbol{\Psi}^2\right\} dv - \int\left\{\boldsymbol{R}\boldsymbol{\Psi} + \boldsymbol{Q}\boldsymbol{\Phi}\right\} dv \\ \quad + \frac{1}{2}\int\left\{\boldsymbol{Q}^2 - \vec{\boldsymbol{P}}^2\right\} dv. \end{array}\right\} \tag{26}$$

From the point of view of the derivation of the field equations (21), the addition to the Hamiltonian of the last integral, which is an invariant function of the variables of the heavy particles, remains of course arbitrary. The reason for its inclusion in the present case, in contrast to the omission of the corresponding integral $\frac{1}{2}\int\left\{(\vec{\boldsymbol{T}})^2-(\vec{\boldsymbol{S}})^2\right\}dv$ in the vector meson theory, will become apparent in the next section.

The static parts $\boldsymbol{\Psi}^\circ$, $\boldsymbol{\Phi}^\circ$ of the pseudoscalar meson fields will be defined as the solution of the equations

$$\boldsymbol{\Phi}^\circ = 0, \tag{27}$$

$$\left\{\begin{array}{l} \vec{\boldsymbol{\Gamma}}^\circ = -\operatorname{grad}\boldsymbol{\Psi}^\circ + \vec{\boldsymbol{P}} \\ \operatorname{div}\vec{\boldsymbol{\Gamma}}^\circ + \kappa^2\boldsymbol{\Psi}^\circ = 0; \end{array}\right\} \tag{28}$$

the equations (28) are equivalent to

$$\Delta\boldsymbol{\Psi}^\circ - \kappa^2\boldsymbol{\Psi}^\circ = \operatorname{div}\vec{\boldsymbol{P}}, \tag{29}$$

giving

$$\boldsymbol{\Psi}^\circ = -\int \operatorname{div}\vec{\boldsymbol{P}}\left(\vec{x}'\right)\cdot\varphi(r)\,dv'. \tag{30}$$

Defining new field variables $\boldsymbol{\Psi}^1$, $\boldsymbol{\Phi}^1$ by the relations

$$\left\{\begin{array}{l} \boldsymbol{\Psi} = \boldsymbol{\Psi}^\circ + \boldsymbol{\Psi}^1 \\ \boldsymbol{\Phi} = \phantom{\boldsymbol{\Psi}^\circ +{}} \boldsymbol{\Phi}^1, \end{array}\right\} \tag{31}$$

we obtain a separation of the Hamiltonian

$$\mathcal{H}_\Phi = \mathcal{H}_\Phi{}^\circ + \mathcal{H}_\Phi{}^1 + \mathcal{U}_\Phi \tag{32}$$

entirely analogous to (16), since the cross-terms again vanish on account of (28). We have here

$$\mathcal{H}_{\Phi}{}^{\circ} = \frac{1}{2}\int\left\{\left(\vec{\boldsymbol{\Gamma}}^{\circ}\right)^{2} + \kappa^{2}\,(\boldsymbol{\Psi}^{\circ})^{2} - \left(\vec{\boldsymbol{P}}\right)^{2}\right\} dv, \tag{33}$$

$$\mathcal{H}_{\Phi}{}^{1} = \frac{1}{2}\int\left\{(\boldsymbol{\Phi}^{1})^{2} + (\text{grad}\,\boldsymbol{\Psi}^{1})^{2} + \kappa^{2}\,(\boldsymbol{\Psi}^{1})^{2}\right\} dv, \tag{34}$$

$$\mathcal{W}_{\Phi} = -\int\left\{\boldsymbol{R}\,(\boldsymbol{\Psi}^{\circ} + \boldsymbol{\Psi}^{1}) + \boldsymbol{Q}\,\boldsymbol{\Phi}^{1}\right\} dv + \frac{1}{2}\int \boldsymbol{Q}^{2}\, dv. \tag{35}$$

If we again provisorily look apart from the non-commutability of spin and isotopic spin matrices, we see that this separation is effected by applying to the total Hamiltonian $\mathcal{H}_{k} + \mathcal{H}_{\Phi}$ a canonical transformation, the operator of which is

$$\mathcal{S} = e^{\frac{i}{\hbar c}\mathcal{K}_{\Phi}},$$

with

$$\mathcal{K}_{\Phi} = \int \boldsymbol{\Psi}^{\circ}\,\boldsymbol{\Phi}\, dv. \tag{36}$$

The static interaction is in this case given by (33).

4. Calculation of the static interaction potentials.

It remains to put the static interactions derived in the preceding sections into the form of potential energies of the heavy particles, i. e. to express them as explicit functions of the dynamical variables of these particles. Let us begin with the expression (17) relative to the vector meson theory. By partial integrations and use of the equations (9) and (10), we find readily

$$\mathcal{H}_{F}^{\circ} = \frac{1}{2}\int\left\{\boldsymbol{V}^{\circ}\,\boldsymbol{N} + \vec{\boldsymbol{G}}^{\circ}\,\vec{\boldsymbol{S}}\right\} dv;$$

with the help of (14) and (13), we obtain further* from the definition (10)

$$\left.\begin{aligned}\vec{G}^\circ &= \vec{S} - \operatorname{rot}\int\left(\vec{S}(\vec{x}')\wedge\operatorname{grad}'\varphi\right)dv' \\ &= \vec{S} + \int\vec{S}(\vec{x}')\,\Delta'\varphi\,dv' + \int\left(\vec{S}(\vec{x}')\cdot\operatorname{grad}'\right)\operatorname{grad}\varphi\,dv' \\ &= \kappa^2\int\vec{S}(\vec{x}')\,\varphi\,dv' + \int\left(\vec{S}(\vec{x}')\cdot\operatorname{grad}'\right)\operatorname{grad}\varphi\,dv',\end{aligned}\right\}\quad(37)$$

so that we get

$$\left.\begin{aligned}\mathcal{H}_F^\circ &= \frac{1}{2}\int N(\vec{x})\,N(\vec{x}')\,\varphi(r)\,dv\,dv' \\ &+ \frac{\kappa^2}{2}\int\vec{S}(\vec{x})\,\vec{S}(\vec{x}')\,\varphi(r)\,dv\,dv' \\ &+ \frac{1}{2}\int\left(\vec{S}(\vec{x})\operatorname{grad}\right)\left(\vec{S}(\vec{x}')\operatorname{grad}'\right)\varphi(r)\,dv\,dv',\end{aligned}\right\}\quad(38)$$

and finally, introducing through (3) and (4) the variables of the heavy particles,

$$\left.\begin{aligned}\mathcal{H}_F^\circ = \frac{1}{2}\sum_{i,k}{}'\left(\overset{(i)}{\tau}\,\overset{(k)}{\tau}\right)\cdot\Big\{&g_1^2 + g_2^2\,\overset{(i)}{\rho_3}\,\overset{\rightarrow(i)}{\sigma}\,\overset{(k)}{\rho_3}\,\overset{\rightarrow(k)}{\sigma} \\ +\left(\frac{g_2}{\kappa}\right)^2\left(\overset{(i)}{\rho_3}\,\overset{\rightarrow(i)}{\sigma}\,\overset{(i)}{\operatorname{grad}}\right)&\left(\overset{(k)}{\rho_3}\,\overset{\rightarrow(k)}{\sigma}\,\overset{(k)}{\operatorname{grad}}\right)\Big\}\,\varphi\left(\overset{(ik)}{r}\right), \\ \left(\overset{(ik)}{r} = \left|\overset{\rightarrow(i)}{x} - \overset{\rightarrow(k)}{x}\right|\right)&.\end{aligned}\right\}\quad(39)$$

If we put all $\overset{(i)}{\rho_3} = 1$, this expression gives the potential energy of static interaction due to the vector meson fields, including—as a defect inevitable in any theory treating the nuclear particles as material points—the infinite static self-energies of these particles.

* When necessary, we affect the operators grad, div, rot ... with the same index as the point at which the derivatives are to be taken.

Passing now to the case of the pseudoscalar meson theory, we get similarly from (33) and (28)

$$\mathscr{H}^{\circ}_{\Phi} = \frac{1}{2}\int \left\{ \vec{\Gamma}^{\circ} \cdot \vec{P} - \left(\vec{P}\right)^2 \right\} dv ,$$

and, with (30),

$$\left.\begin{aligned} \vec{\Gamma}^{\circ} &= \vec{P} - \text{grad} \int \vec{P}\left(\vec{x}'\right) \text{grad}' \,\varphi \, dv' \\ &= \vec{P} - \int \left(\vec{P}\left(\vec{x}'\right) \cdot \text{grad}'\right) \text{grad}\, \varphi \, dv', \end{aligned}\right\} \tag{40}$$

so that the static interaction in this case becomes

$$\mathscr{H}^{\circ}_{\Phi} = -\frac{1}{2}\int \left(\vec{P}\left(\vec{x}\right) \text{grad}\right) \left(\vec{P}\left(\vec{x}'\right) \text{grad}'\right) \varphi\,(r)\, dv\, dv' \tag{41}$$

or, on account of (24),

$$\mathscr{H}^{\circ}_{\Phi} = -\frac{1}{2}\sum_{i,k}{}' \left(\boldsymbol{\tau}^{(i)} \boldsymbol{\tau}^{(k)}\right) \left(\frac{f_2}{\kappa}\right)^2 \left(\vec{\sigma}^{(i)} \text{grad}^{(i)}\right) \left(\vec{\sigma}^{(k)} \text{grad}^{(k)}\right) \varphi \left(r^{(ik)}\right). \tag{42}$$

It will be observed that, if the invariant $\frac{1}{2}\int \left\{ \boldsymbol{\Omega}^2 - \left(\vec{P}\right)^2 \right\} dv$ had not been included in the Hamiltonian $\mathscr{H}_{\Phi}$ defined by (26), we would have obtained a supplementary static interaction

$$\frac{1}{2}\int \left(\vec{P}\right)^2 dv = \frac{1}{2}\sum_{i,k}{}' \left(\boldsymbol{\tau}^{(i)} \boldsymbol{\tau}^{(k)}\right) \left(\vec{\sigma}^{(i)} \vec{\sigma}^{(k)}\right) \delta \left(\vec{x}^{(i)} - \vec{x}^{(k)}\right),$$

which, as discussed by Kemmer[18)], is of so strongly singular a character that it could not give any finite binding energy for the deuteron. With our choice of the Hamiltonian $\mathscr{H}_{\Phi}$ this singular term is eliminated, and there occurs instead a term $\frac{1}{2}\int \boldsymbol{\Omega}^2 dv$ which, though of the same type, is only of the second order in the velocities and therefore inseparable from all other infinite terms to be discarded according to the correspondence point of view developed in Part II. In the vector meson theory, no such singular terms occur if, following YUKAWA[17], the Hamiltonian (7) is adopted in preference to an expression differing from (7) by the term $\frac{1}{2}\int\left\{\left(\vec{\boldsymbol{T}}\right)^2-\left(\vec{\boldsymbol{S}}\right)^2\right\}dv$, which has been considered by other authors[17]. This would perhaps appear as a natural way of removing the arbitrariness connected with the occurrence in the formalism of such singular terms of direct coupling between the nuclear particles.

For the two remaining types of meson fields, we shall only write down the expressions for the static interactions, resulting from entirely similar considerations:

Scalar meson field:

$$-\frac{1}{2}\sum_{i,k}{}'\left(\boldsymbol{\tau}^{(i)}\boldsymbol{\tau}^{(k)}\right)f_1'^2\,\varphi\left(r^{(ik)}\right). \tag{43}$$

Pseudovector meson field:

$$\left.\begin{aligned}&-\frac{1}{2}\sum_{i,k}{}'\left(\boldsymbol{\tau}^{(i)}\boldsymbol{\tau}^{(k)}\right)\Big\{g_1'^2\left(\vec{\sigma}^{(i)}\vec{\sigma}^{(k)}\right)\\&+\left[\left(\frac{g_1'}{\kappa}\right)^2-\left(\frac{g_2'}{\kappa}\right)^2\right]\left(\vec{\sigma}^{(i)}\operatorname{grad}^{(i)}\right)\left(\vec{\sigma}^{(k)}\operatorname{grad}^{(k)}\right)\Big\}\,\varphi\left(r^{(ik)}\right).\end{aligned}\right\} \tag{44}$$

Comparing the expressions (39), (42), (43), (44), we see that they contain three different kinds of potentials, viz. (apart from the dependence on isotopic spin common to all of them): 1) a potential $\varphi(r^{(ik)})$ depending only on the mutual distances of the nuclear particles; 2) a spin-spin coupling $\left(\vec{\sigma}^{(i)}\vec{\sigma}^{(k)}\right)\varphi\left(r^{(ik)}\right)$; 3) a directional coupling

$$\frac{1}{\kappa^2}\left(\vec{\sigma}^{(i)}\operatorname{grad}^{(i)}\right)\left(\vec{\sigma}^{(k)}\operatorname{grad}^{(k)}\right)\varphi\left(r^{(ik)}\right)$$

which, apart from the sign, is of the type of a dipole interaction. The most general form of static interaction, resulting from an arbitrary mixture of the four types of meson fields is, therefore, a linear combination of these three kinds of potentials,

$$\left.\begin{aligned}\frac{1}{2}\sum_{i,k}{}'\left(\boldsymbol{\tau}^{(i)}\boldsymbol{\tau}^{(k)}\right)\Big\{G_1+G_2\left(\vec{\sigma}^{(i)}\vec{\sigma}^{(k)}\right)\\+\frac{G_3}{\kappa^2}\left(\vec{\sigma}^{(i)}\operatorname{grad}^{(i)}\right)\left(\vec{\sigma}^{(k)}\operatorname{grad}^{(k)}\right)\Big\}\,\varphi\left(r^{(ik)}\right),\end{aligned}\right\}\tag{45}$$

with coefficients given by

$$\left\{\begin{aligned}G_1&=g_1^2-f_1'^2\\G_2&=g_2^2-g_1'^2\\G_3&=g_2^2-f_2'^2-g_1'^2+g_2'^2.\end{aligned}\right\}\tag{46}$$

Part II.

Limitations of the Formalism.

1. Difficulties arising from the potential of dipole interaction type.

For the fixation of the choice between the four possible types of meson fields responsible for the nuclear forces, an important criterion is afforded by our empirical knowledge of the stationary states of the deuteron. In trying to account for the deuteron spectrum by means of one type of meson fields only, vector meson fields have generally been adopted[17], in spite of the fact that the corresponding static interaction (39) includes a term of dipole interaction type which, strictly speaking, makes the existence of stationary states of finite binding energy impossible. In view of the provisory character of any quantum field theory at the present stage, this difficulty might, in fact, not be deemed fundamental, and it might be attempted to avoid it by a cutting-off prescription. Such an attempt, carried out by Bethe[10], has led to the conclusion that, while the way in which the cutting-off is performed is of small influence on the results, the value to be assumed for the cutting-off radius depends critically on the combination of charged and neutral meson fields adopted: if one uses the symmetrical combination proposed by Kemmer, the cutting-off radius should be chosen larger than the range of the nuclear forces, and a reasonable value of this radius could only be obtained if the meson field were assumed to be purely neutral. Even looking apart from the unsatisfactory character, pointed out in the Introduction, of a purely neutral meson theory of nuclear forces, any meson theory involving forces of the dipole interaction type is affected, however, as we shall now proceed to show, by a more serious difficulty, due to the non-static effects connected with such forces.

In order to get an idea of the nature of these non-static effects, we have, according to the considerations of section 2 of Part I, to examine the time-derivatives of the dynamical variables of the nuclear particles occurring in the expression of the transformation matrix $\mathcal{S}$, i. e. $\vec{x}^{(i)}$, $\vec{\sigma}^{(i)}$, $\rho_3^{(i)}$, $\boldsymbol{\tau}^{(i)}$; neglecting all terms depending on the velocities of the heavy particles, we are left with the consideration of $\dot{\vec{\sigma}}^{(i)}$ and $\dot{\boldsymbol{\tau}}^{(i)}$. The motions corresponding to these time-variations may be described as a precession of the vector $\vec{\sigma}^{(i)}$ in ordinary space and of the vector $\boldsymbol{\tau}^{(i)}$ in symbolic space; for instance, in the most familiar case of the vector meson theory, developed in Section I of Part I, these motions are defined, to the approximation indicated, by the equations*

$$\left.\begin{aligned} \dot{\vec{\sigma}}^{(i)} &= \frac{1}{\hbar c}\,\frac{g_2}{\kappa}\,\boldsymbol{\tau}^{(i)}\left[\vec{\boldsymbol{G}}\left(\vec{x}^{(i)}\right)\wedge\vec{\sigma}^{(i)}-\vec{\sigma}^{(i)}\wedge\vec{\boldsymbol{G}}\left(\vec{x}^{(i)}\right)\right],\\ \dot{\boldsymbol{\tau}}^{(i)} &= \frac{1}{\hbar c}\left\{\left[g_1\boldsymbol{V}\left(\vec{x}^{(i)}\right)+\frac{g_2}{\kappa}\vec{\boldsymbol{G}}\left(\vec{x}^{(i)}\right)\cdot\vec{\sigma}^{(i)}\right]\boldsymbol{\wedge}\boldsymbol{\tau}^{(i)}+\text{conj.}\right\}\end{aligned}\right\}\quad(47)$$

* The symbols $\wedge$ and $\boldsymbol{\wedge}$ indicate a vector product in ordinary and symbolic space respectively. The abbreviation "conj." will sometimes be used to represent the Hermitian conjugate of the expression preceding it.

(In the last formula, a term proportional to the small quantity $\frac{(M_N - M_P)\,c}{\hbar\kappa}$ has been neglected). A criterion of the importance of the non-static effects can be derived from an investigation of the amount of these precessions which is due to the static part of the meson fields. In our example of the vector meson theory, we obtain the corresponding time-variations by inserting in (47) the expressions (14) and (37) of $\boldsymbol{V}^\circ$ and $\vec{\boldsymbol{G}}^\circ$:

$$\left.\begin{aligned}
\dot{\vec{\sigma}}^{(i)} &= \Big\{ \frac{2}{\hbar c} \left(\frac{g_2}{\kappa}\right)^2 \sum_{i \neq k}' \left(\boldsymbol{\tau}^{(i)} \boldsymbol{\tau}^{(k)}\right) \Big[\kappa^2 \vec{\sigma}^{(k)} \\
&\quad + \left(\vec{\sigma}^{(k)} \operatorname{grad}^{(k)}\right) \operatorname{grad}^{(i)}\Big]\, \varphi\left(r^{(ik)}\right) \Big\} \wedge \vec{\sigma}^{(i)}, \\
\dot{\boldsymbol{\tau}}^{(i)} &= \Big\{ \frac{2}{\hbar c} \sum_{i \neq k}' \Big[g_1^2 + \left(\frac{g_2}{\kappa}\right)^2 \Big(\kappa^2 \vec{\sigma}^{(k)} \cdot \vec{\sigma}^{(i)} \\
&\quad + \left(\vec{\sigma}^{(k)} \operatorname{grad}^{(k)}\right) \left(\vec{\sigma}^{(i)} \operatorname{grad}^{(i)}\right)\Big)\Big]\, \varphi\left(r^{(ik)}\right) \boldsymbol{\tau}^{(k)} \Big\} \wedge \boldsymbol{\tau}^{(i)},
\end{aligned}\right\} \quad (48)$$

the terms corresponding to $k = i$ in the summations vanishing automatically. On account of the separation (16) of the Hamiltonian, these equations are of course simply

$$\dot{\vec{\sigma}}^{(i)} = \frac{i}{\hbar c}\left[\mathscr{H}_F^\circ, \vec{\sigma}^{(i)}\right], \quad \dot{\boldsymbol{\tau}}^{(i)} = \frac{i}{\hbar c}\left[\mathscr{H}_F^\circ, \boldsymbol{\tau}^{(i)}\right],$$

with $\mathscr{H}_F^\circ$ given by (39); and, for the most general form of meson theory, the corresponding equations may be obtained in the same way, $\mathscr{H}_F^\circ$ being replaced by the general expression (45):

$$\left.\begin{aligned}\dot{\vec{\sigma}}^{(i)} &= \left\{\frac{2}{\hbar c}\sum_{i \neq k}{}' \left(\boldsymbol{\tau}^{(i)}\boldsymbol{\tau}^{(k)}\right)\left[G_2 \vec{\sigma}^{(k)}\right.\right. \\ &\left.\left.+ \frac{G_3}{\kappa^2}\left(\vec{\sigma}^{(k)}\operatorname{grad}^{(k)}\right)\operatorname{grad}^{(i)}\right]\varphi\left(r^{(ik)}\right)\right\}\Lambda\vec{\sigma}^{(i)} \\ \dot{\boldsymbol{\tau}}^{(i)} &= \left\{\frac{2}{\hbar c}\sum_{i \neq k}{}' \left[G_1 + G_2\left(\vec{\sigma}^{(i)}\vec{\sigma}^{(k)}\right)\right.\right. \\ &\left.\left.+ \frac{G_3}{\kappa^2}\left(\vec{\sigma}^{(k)}\operatorname{grad}^{(k)}\right)\left(\vec{\sigma}^{(i)}\operatorname{grad}^{(i)}\right)\right]\varphi\left(r^{(ik)}\right)\boldsymbol{\tau}^{(k)}\right\}\boldsymbol{\Lambda}\boldsymbol{\tau}^{(i)}.\end{aligned}\right\} \quad (49)$$

Now, it is clear that if the periods of such precessions are not large compared with the time of propagation of the main part of the non-static meson fields through a distance of the order of the range of the static nuclear forces, there is no justification in using only the static parts of the meson fields for the determination of the stationary states of a nuclear system. According to (49), the angular velocity of precession of spin and isotopic spin of a nuclear particle is, for sufficiently small values of the distance r of the next neighbour in the nuclear configuration considered, of the order of magnitude $\frac{1}{\hbar}\frac{G_3}{\kappa^2}\frac{1}{4\pi r^3}$ if $G_3 \neq 0$, and $\frac{1}{\hbar}G\frac{1}{4\pi r}$ if $G_3 = 0$, G being written for G_1 or G_2; on the other hand, the time of propagation of the main part of the non-static meson fields through the distance κ^{-1} will be of the same order of magnitude as $\frac{1}{\kappa c}$. The condition just formulated defines therefore a critical value r_c of the distance r, such that the consideration of the non-static forces will be important as soon as $r < r_c$. If the theory involves static couplings of the dipole interaction type, we have therefore

$$\kappa r_c = \sqrt[3]{\frac{G_3}{4\pi\hbar c}}, \quad (50)$$

while, if no such couplings occur, we find for the critical distance the smaller value

$$\kappa r_c = \frac{G}{4\pi \hbar c}. \tag{51}$$

In a purely neutral meson theory, entirely similar considerations apply to the $\vec{\sigma}$-precession. In particular, taking the theory of neutral vector mesons investigated by BETHE[10], for which he gives*

$$\frac{G_3}{4\pi \hbar c} = \frac{g_2^2}{4\pi \hbar c} = 0.08\,,$$

we see that the critical distance given by (50) is of the same order of magnitude as the cutting-off radius

$$\kappa r_0 = 0.320 \text{ or } 0.436.$$

It would, therefore, not seem consistent to disregard the non-static forces even in a treatment involving a cutting-off prescription. Neither can there be any hope that an explicit consideration of the precession effects just discussed would permit to avoid the singularities of dipole interaction type, since the contributions to the energy of the system arising from these effects are of an essentially different form. Above all, however, such large precession effects, although not directly depending on the quantization of the meson fields, could not be unambiguously separated from the typical quantum effects which give rise to the well-known divergences of any theory of quantized fields. We may, therefore, conclude that, within the frame of the present formalism, we can only expect to obtain a meson

* The explicit appearance of the factor 4π is due to our use of units analogous to the Heaviside units of electrodynamics.

field theory capable of consistent interpretation (to a sufficiently restricted extent) if all couplings of dipole interaction type are eliminated at the outset, i. e. if

$$G_3 = 0. \tag{52}$$

On the other hand, a more detailed examination of the non-static meson fields due to the spin and isotopic spin precessions, to which we will come back in section 5, shows that, in a theory which does not involve any coupling of dipole interaction type, the effect of these fields on the stationary states of a nuclear system will actually be much smaller than that of the static forces, if the mean distance between any pair of nuclear particles in such a state is larger than the critical distance defined by (51). A comparison with the empirical data, which will be given later in connection with the discussion of the properties of the deuteron, shows that the last condition is well fulfilled for ordinary nuclear systems. As regards the difficulties of field quantization, we might perhaps expect that the unambiguous conclusions derived by completely disregarding them would still be reliable provided the theory using the unquantized fields does not itself contain any ambiguity. From this point of view, we should conclude that, in a meson theory satisfying, besides (52), the condition just discussed, only the static potential will be of importance for the determination of the stationary states of nuclear systems.

2. Choice of a special form of meson theory.

Let us now consider the different forms of the meson theory satisfying the requirements in question. In order to secure agreement with the known properties of the stationary states of the deuteron, we have to impose further restrictions on the static potentials given by these theories. In particular, we shall require that the static potential be attractive in the ^3S and ^1S states of the deuteron, revealed by scattering experiments, and that the ^3S-level be lower than the ^1S-level. It will be seen—always assuming KEMMER's symmetrical combination of charged and neutral meson fields—that these simple qualitative requirements, which lead to two independent inequalities involving G_1 and G_2, are, together with (52), sufficient to restrict the choice of the form of meson theory to an essentially unique possibility.

According to the expression (45) of the static potential, with $G_3 = 0$, the mentioned inequalities to be fulfilled by G_1 and G_2 are

$$-3(G_1 + G_2) < G_1 - 3G_2 < 0,$$

which reduce to

$$\left.\begin{aligned} G_1 &> 0, \\ G_2 &> \frac{G_1}{3}. \end{aligned}\right\} \tag{53}$$

From the values (46) of G_1 and G_2 it is immediately apparent that the conditions (52), (53) cannot be fulfilled by any theory involving only one of the four possible types of meson fields, so that we are led to consider the possible mixtures or "compositions" of two or more of these types of fields. It is then easily verified that, if we try to compose only two types of fields, the only possibility is a mixture of vector and pseudoscalar meson fields satisfying the conditions

$$f_2^2 = g_2^2, \tag{54}$$

$$g_2^2 > \frac{1}{3} g_1^2. \tag{55}$$

It should further be observed that from the point of view of nuclear forces the possible compositions of three or four types of meson fields (obtained by adding to a mixture of vector and pseudoscalar fields either a pseudovector field, or a scalar field, or both) only differ from the composition of two fields just mentioned by unessential numerical changes of the constants g, f, so that their greater complication is not compensated by any advantage*. We shall therefore in the following adopt the simplest mixture of vector and pseudoscalar meson fields, as defined by the relations (54), (55).

The corresponding Hamiltonian may be written

$$\mathcal{H} = \mathcal{H}_k + \mathcal{H}_F + \mathcal{H}_\Phi. \tag{56}$$

with $\mathcal{H}_k$, $\mathcal{H}_F$, $\mathcal{H}_\Phi$ given by (6), (7) and (26), respectively; the commutation rules between pairs of canonically conjugated variables are given by (8) and (25), all other pairs of variables commuting. The separation of the static potential may be effected by the canonical transformation defined by the operator

$$\left\{\begin{aligned} & \mathcal{S} = e^{\frac{i}{\hbar c}\mathcal{K}} \\ \text{with}\quad & \mathcal{K} = \mathcal{K}_F + \mathcal{K}_\Phi, \end{aligned}\right\} \tag{57}$$

* An entirely similar discussion may be carried out in the case of a purely neutral meson theory, the static potential being then given by the expression (45) with the factors $\left(\boldsymbol{\tau}^{(i)}\,\boldsymbol{\tau}^{(k)}\right)$ omitted. The inequalities to be fulfilled by G_1 and G_2 are in this case

$$G_1 + G_2 < G_1 - 3\,G_2 < 0,$$

reducing to

$$G_1 < 3\,G_2, \quad G_2 < 0.$$

Also in this case, there is one possible composition of two fields, viz. a mixture of a scalar meson field and a pseudovector meson field, and this possibility is essentially unique.

$\mathcal{H}_F$ and $\mathcal{H}_\Phi$ being given by (20) and (36); the new variables $\vec{x}^{(i)\prime} = \vec{x}^{(i)}$, $\vec{p}^{(i)\prime}$, $\rho_3^{(i)\prime} = \rho_3^{(i)}$, $\vec{\sigma}^{(i)\prime}$, $\tau^{(i)\prime}$; $\vec{\boldsymbol{U}}'$, $\vec{\boldsymbol{F}}'$; $\boldsymbol{\Psi}'$, $\boldsymbol{\Phi}' = \boldsymbol{\Phi}$ are defined in terms of the old variables by formulae of the type $A' = \mathcal{S}^{-1} A \mathcal{S}$. As a function of the new variables, the Hamiltonian takes then the form

$$\mathcal{H} = \mathcal{S}' \mathcal{H}' \mathcal{S}'^{-1}, \tag{58}$$

if $\mathcal{H}'$ and $\mathcal{S}'$ denote the same functions of the new variables as the functions $\mathcal{H}$ and $\mathcal{S}$ of the old variables defined by (56) and (57). In the next sections, we shall discuss the general features of the physical interpretation of this formal scheme.

3. Interpretation of the transformed variables.

The interpretation of the different variables is closely connected with the form of the fundamental integrals of the system, representing its total linear momentum*

$$\sum_i \vec{p}^{(i)} + \sum_{\mu=1}^{3} \int \boldsymbol{F}^\mu \operatorname{grad} \boldsymbol{U}^\mu \, dv - \int \boldsymbol{\Phi} \operatorname{grad} \boldsymbol{\Psi} \, dv, \tag{59}$$

its total angular momentum*

$$\left.\begin{aligned} &\sum_i \left\{ \vec{x}^{(i)} \wedge \vec{p}^{(i)} + \frac{\hbar c}{2} \vec{\sigma}^{(i)} \right\} + \sum_{\mu=1}^{3} \int \boldsymbol{F}^\mu (\vec{x} \wedge \operatorname{grad}) \boldsymbol{U}^\mu \, dv \\ &\quad - \int \vec{\boldsymbol{U}} \wedge \vec{\boldsymbol{F}} \, dv - \int \boldsymbol{\Phi} (\vec{x} \wedge \operatorname{grad}) \boldsymbol{\Psi} \, dv, \end{aligned}\right\} \tag{60}$$

and its total electric charge

* The formulae (59) and (60) represent the indicated quantities multiplied by c.

$$e\sum_i \frac{1-\tau_3^{(i)}}{2}+\frac{e}{\hbar c}\int\{\vec{U}\wedge\vec{F}\}_3\,dv-\frac{e}{\hbar c}\int\{\boldsymbol{\Psi}\wedge\boldsymbol{\Phi}\}_3\,dv. \quad (61)$$

The conservation laws for these quantities follow immediately from the invariance properties of the Hamiltonian, since the expressions (59) and (60) are respectively proportional to the operators of the infinitesimal transformations of the groups of translations and rotations in ordinary space, while the expression (61) is closely connected to the component of index **3** of the transformation in isotopic space analogous to a rotation, viz*.

$$\frac{\hbar c}{2}\sum_i \boldsymbol{\tau}^{(i)}-\int\vec{U}\wedge\vec{F}\,dv+\int\boldsymbol{\Psi}\wedge\boldsymbol{\Phi}\,dv \qquad (62)$$

(there being here no terms analogous to orbital momenta).

The three integrals (59), (60), (61) have the property of being sums of terms referring separately to the heavy particles and to each type of meson field. As regards the angular momentum, it is further possible to distinguish, for the heavy particles and the vector meson fields, between orbital momentum and spin, while the pseudoscalar meson fields have of course no spin. It is just these additivity properties which provide the justification for the usual interpretations of the variables. This is first of all the case for the variables $\vec{p}^{(i)}$, $\vec{\sigma}^{(i)}$ and $\boldsymbol{\tau}^{(i)}$ of the heavy particles; the expression (61) shows further how the two first symbolic components of the field variables are associated with charged mesons, while the components of index **3** correspond to neutral mesons; finally, if the linear momentum (59) is expressed in the usual way as a function of the Fourier

* The formula (62) represents the indicated quantity multiplied by c.

amplitudes of the field variables, the resulting expression shows clearly the connection of these amplitudes with mesons of definite momentum.

Now, it is of course desirable that also the transformed variables should possess all the properties just discussed, and it should be pointed out that this is actually the case for the transformation defined by the operator (57). In fact, the invariance of this operator $\mathcal{S}$ with respect to translations and rotations in ordinary space as well as to rotations in symbolic space leads at once to the conclusion that the operators (59), (60), (62) commute with $\mathcal{S}$, so that the integrals of linear momentum, angular momentum and electric charge of the system, in contrast to the energy, are the same functions of the new variables as the functions of the old variables given by (59), (60) and (61).

The requirement that our canonical transformation should thus conserve the form of the integrals (59), (60), (61) restricts to some extent the arbitrariness in the choice of the operator $\mathcal{S}$ as a product of exponential factors (cf. footnote on p. 18). In the first place, all the exponents must be invariant with respect to ordinary rotations; further, in order to uphold the additivity property of the total electric charge (61), they should be invariant with respect to rotations in symbolic space about the "direction" of index **3**, i. e. they should be of the form $A_3\, B_3$, or $A_1\, B_1 + A_2\, B_2$, or $\boldsymbol{A}\,\boldsymbol{B}$. The form (57) for $\mathcal{S}$ has been adopted only on account of its greater symmetry and simplicity.

4. The Hamiltonian in terms of the new variables.

We shall now proceed to derive a more explicit expression of the Hamiltonian (58) in terms of the new variables, bringing out the effects due to the non-commutability of the spin and isotopic spin variables. For this purpose, we shall start by replacing in (56) $\mathcal{H}_F$ and $\mathcal{H}_\Phi$ by their expressions (16) and (32) resulting from the explicit introduction of the static parts of the meson fields. Neglecting all terms of higher order than the first in the velocities of the heavy particles, we may thus write

$$\mathcal{H} = \mathcal{H}_n + \mathcal{H}_f\left(\vec{\boldsymbol{U}}^1, \vec{\boldsymbol{F}}^1; \boldsymbol{\Psi}^1, \boldsymbol{\Phi}^1\right) + \mathcal{W}, \qquad (63)$$

where the first term

$$\mathcal{H}_n = \mathcal{H}_k + \mathcal{V}_n \qquad (64)$$

is the sum of the kinetic energy $\mathcal{H}_k$ and the static interaction

$$\mathcal{V}_n = \frac{1}{2} \sum_{i,k}{}' \left(\boldsymbol{\tau}^{(i)} \boldsymbol{\tau}^{(k)}\right) \left[g_1^2 + g_2^2 \left(\vec{\sigma}^{(i)} \vec{\sigma}^{(k)}\right)\right] \varphi\left(r^{(ik)}\right) \qquad (65)$$

of the system of nuclear particles, the second term is the function

$$\left.\begin{aligned} \mathcal{H}_f = \frac{1}{2} \int \left\{ \vec{\boldsymbol{F}}^2 + \kappa^{-2} \left(\operatorname{div} \vec{\boldsymbol{F}}\right)^2 + \left(\operatorname{rot} \vec{\boldsymbol{U}}\right)^2 + \kappa^2 \vec{\boldsymbol{U}}^2 \right\} dv \\ + \frac{1}{2} \int \left\{ \boldsymbol{\Phi}^2 + (\operatorname{grad} \boldsymbol{\Psi})^2 + \kappa^2 \boldsymbol{\Psi}^2 \right\} dv \end{aligned}\right\} \qquad (66)$$

representing the Hamiltonian of a system of pure meson fields, taken for the field quantities $\vec{\boldsymbol{U}}^1, \vec{\boldsymbol{F}}^1; \boldsymbol{\Psi}^1, \boldsymbol{\Phi}^1$ defined by (15) and (31), and the third term is the coupling

$$\mathcal{W} = -\int \left\{ \vec{\boldsymbol{M}} \vec{\boldsymbol{U}} + \vec{\boldsymbol{T}} \vec{\boldsymbol{F}} + \boldsymbol{R} \boldsymbol{\Psi} + \boldsymbol{Q} \boldsymbol{\Phi} \right\} dv, \qquad (67)$$

of the first order in the velocities.

Now, if A is any function of the old variables, we have for its expression in terms of the new variables the general formula

$$\left.\begin{aligned} A = \mathcal{S}' A' \mathcal{S}'^{-1} &= A' + \frac{i}{\hbar c}[\mathcal{K}', A'] + \cdots + \frac{1}{l!}\left\{\frac{i}{\hbar c}\mathcal{K}', A'\right\}^l + \cdots, \\ \text{with } \left\{\frac{i}{\hbar c}\mathcal{K}', A'\right\}^l &= \underbrace{\frac{i}{\hbar c}\left[\mathcal{K}', \frac{i}{\hbar c}\left[\mathcal{K}', \cdots, \frac{i}{\hbar c}[\mathcal{K}', A']\right]\cdots\right]}_{(l \text{ brackets})}, \end{aligned}\right\} \quad (68)$$

in which A' denotes in the usual way the same function of the new variables as the function A of the old variables. We therefore get

$$\left.\begin{aligned} \mathcal{H} = \mathcal{H}_n' + \mathcal{H}_f\left(\vec{\boldsymbol{U}}' - \vec{\boldsymbol{u}}', \vec{\boldsymbol{F}} - \vec{\boldsymbol{f}}'; \boldsymbol{\Psi}' - \boldsymbol{\psi}', \boldsymbol{\Phi}'\right) + \mathcal{W}' \\ + \sum_{l=1}^{\infty} \frac{1}{l!}\left\{\frac{i}{\hbar c}\mathcal{K}', \mathcal{H}_n'\right\}^l + \sum_{l=1}^{\infty}\frac{1}{l!}\left\{\frac{i}{\hbar c}\mathcal{K}', \mathcal{W}'\right\}^l, \end{aligned}\right\} \quad (69)$$

with

$$\vec{\boldsymbol{u}}' = \sum_{l=0}^{\infty}\frac{1}{l!}\left\{\frac{i}{\hbar c}\mathcal{K}', \vec{\boldsymbol{U}}^{0\prime}\right\}^l - \sum_{l=1}^{\infty}\frac{1}{l!}\left\{\frac{i}{\hbar c}\mathcal{K}', \vec{\boldsymbol{U}}'\right\}^l \quad (70)$$

and similar formulae for $\vec{\boldsymbol{f}}'$ and $\boldsymbol{\psi}'$. Since we shall in the following make use exclusively of the new variables, we may from now on, for convenience, omit the primes by which they were hitherto distinguished from the old variables.

Noting that

$$\frac{i}{\hbar c}[\mathcal{K}, \vec{\boldsymbol{U}}] = \vec{\boldsymbol{U}}^\circ, \quad \frac{i}{\hbar c}[\mathcal{K}, \vec{\boldsymbol{F}}] = \vec{\boldsymbol{F}}^\circ, \quad \frac{i}{\hbar c}[\mathcal{K}, \boldsymbol{\Psi}] = \boldsymbol{\Psi}^\circ, \quad (71)$$

we find from (70) for $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{f}}$, $\boldsymbol{\psi}$ expressions of the type

$$\vec{\boldsymbol{u}} = \sum_{l=1}^{\infty}\frac{l}{(l+1)!}\left\{\frac{i}{\hbar c}\mathcal{K}, \vec{\boldsymbol{U}}^\circ\right\}^l, \quad (72)$$

showing that, as was to be expected, these quantities would vanish if we could look apart from the non-commutability of all spin variables. It is further apparent that $\vec{\boldsymbol{u}}, \vec{\boldsymbol{f}}, \boldsymbol{\psi}$ satisfy equations of the same form as the equations (9), (11), (27), (29) for $\vec{\boldsymbol{U}}^{\circ}, \vec{\boldsymbol{F}}^{\circ}, \boldsymbol{\psi}^{\circ}$. In fact, we have

$$\left.\begin{aligned} \operatorname{grad}\operatorname{div}\vec{\boldsymbol{f}} - \kappa^2 \vec{\boldsymbol{f}} &= \operatorname{grad} \boldsymbol{n} \\ -\operatorname{rot}\operatorname{rot}\vec{\boldsymbol{u}} - \kappa^2 \vec{\boldsymbol{u}} &= \operatorname{rot}\vec{\boldsymbol{s}} \\ \operatorname{div}\operatorname{grad}\boldsymbol{\psi} - \kappa^2 \boldsymbol{\psi} &= \operatorname{div}\vec{\boldsymbol{p}}, \end{aligned}\right\} \tag{73}$$

with

$$\left.\begin{aligned} \boldsymbol{n} &= \sum_{l=1}^{\infty} \frac{l}{(l+1)!}\left\{\frac{i}{\hbar c}\mathcal{K}, \boldsymbol{N}\right\}^l, \\ \vec{\boldsymbol{s}} &= \sum_{l=1}^{\infty} \frac{l}{(l+1)!}\left\{\frac{i}{\hbar c}\mathcal{K}, \vec{\boldsymbol{S}}\right\}^l \\ \vec{\boldsymbol{p}} &= \sum_{l=1}^{\infty} \frac{l}{(l+1)!}\left\{\frac{i}{\hbar c}\mathcal{K}, \vec{\boldsymbol{P}}\right\}^l. \end{aligned}\right\} \tag{74}$$

In particular, we can derive from (73) the expressions

$$\left.\begin{aligned} \operatorname{div}\vec{\boldsymbol{f}} &= -\int \varphi(r)\, dv' \operatorname{div}\operatorname{grad}\boldsymbol{n}(\vec{x}') = -\int \boldsymbol{n}(\vec{x}')\,\Delta\varphi\, dv' \\ \operatorname{rot}\vec{\boldsymbol{u}} &= -\int \varphi(r)\, dv' \operatorname{rot}\operatorname{rot}\vec{\boldsymbol{s}}(\vec{x}') \\ &= \int \vec{\boldsymbol{s}}(\vec{x}')\,\Delta\varphi \cdot dv' - \int \left(\vec{\boldsymbol{s}}(\vec{x}')\cdot\operatorname{grad}\right)\operatorname{grad}\varphi\, dv' \\ \operatorname{grad}\boldsymbol{\psi} &= -\int \varphi(r)\, dv' \operatorname{grad}\operatorname{div}\vec{\boldsymbol{p}}(\vec{x}') \\ &= -\int \left(\vec{\boldsymbol{p}}(\vec{x}')\operatorname{grad}\right)\operatorname{grad}\varphi\, dv'. \end{aligned}\right\} \tag{75}$$

Taking account of the formulae (73) and (75), we obtain, according to (66), by means of partial integrations,

$$\left.\begin{aligned}\mathcal{H}_f\left(\vec{\boldsymbol{U}}-\vec{\boldsymbol{u}},\ \vec{\boldsymbol{F}}-\vec{\boldsymbol{f}};\ \boldsymbol{\Psi}-\boldsymbol{\psi},\boldsymbol{\Phi}\right) &= \mathcal{H}_f\left(\vec{\boldsymbol{U}},\vec{\boldsymbol{F}};\ \boldsymbol{\Psi},\boldsymbol{\Phi}\right)\\ &+\frac{1}{2}\int\left\{-\kappa^{-2}\boldsymbol{n}\operatorname{div}\vec{\boldsymbol{F}}+\vec{\boldsymbol{s}}\operatorname{rot}\vec{\boldsymbol{U}}-\vec{\boldsymbol{p}}\operatorname{grad}\boldsymbol{\Psi}\right\}dv+\text{conj.}\\ &-\frac{1}{2}\int dv\,dv'\left(\kappa^{-2}\boldsymbol{n}(\vec{x})\,\boldsymbol{n}(\vec{x'})+\vec{\boldsymbol{s}}(\vec{x})\,\vec{\boldsymbol{s}}(\vec{x'})\right)\Delta\varphi;\end{aligned}\right\}\quad(76)$$

besides the written terms in (76), there occurs a further expression

$$\int dv\,dv'\left\{\left(\vec{\boldsymbol{s}}(\vec{x})\operatorname{grad}\right)\left(\vec{\boldsymbol{s}}(\vec{x'})\operatorname{grad}\right)-\left(\vec{\boldsymbol{p}}(\vec{x})\operatorname{grad}\right)\left(\vec{\boldsymbol{p}}(\vec{x'})\operatorname{grad}\right)\right\}\varphi \quad (77)$$

which, however, vanishes except for terms of at least second order in the velocities.

Turning to the fourth term in (69), we may transform it into

$$\sum_{l=1}^{\infty}\frac{1}{l!}\left\{\frac{i}{\hbar c}\mathcal{K},\mathcal{H}_n\right\}^l = -\overset{\odot}{\mathcal{K}} - \sum_{l=1}^{\infty}\frac{1}{(l+1)!}\left\{\frac{i}{\hbar c}\mathcal{K},\overset{\odot}{\mathcal{K}}\right\}^l, \quad (78)$$

if we denote in general by $\overset{\odot}{A}$ the time variation of A

$$\overset{\odot}{A} = \frac{i}{\hbar c}\left[\mathcal{H}_n, A\right] \quad (79)$$

due to the motion of the nuclear particles under the influence of the static forces; thus

$$\overset{\odot}{\mathcal{K}} = \int\left\{\overset{\odot}{\vec{\boldsymbol{F}}}{}^{\circ}\,\vec{\boldsymbol{U}} - \overset{\odot}{\vec{\boldsymbol{U}}}{}^{\circ}\,\vec{\boldsymbol{F}} + \overset{\odot}{\boldsymbol{\Psi}}{}^{\circ}\,\boldsymbol{\Phi}\right\}dv. \quad (80)$$

As regards the last term in (69), we may write, on account of (71),

$$\frac{i}{\hbar c}\left[\mathcal{K},\mathcal{W}\right] = \mathcal{W}_n + w \quad (81)$$

with $$\mathscr{W}_n = -\int \left\{ \vec{M}\,\vec{U}^\circ + \vec{T}\,\vec{F}^\circ + R\,\Psi^\circ \right\} dv \tag{82}$$

and

$$\left.\begin{aligned} w = -\int \Big\{ \frac{i}{\hbar c}\left[\mathscr{K}, \vec{M}\right]\vec{U} + \frac{i}{\hbar c}\left[\mathscr{K}, \vec{T}\right]\vec{F} + \frac{i}{\hbar c}\left[\mathscr{K}, R\right]\Psi \\ + \frac{i}{\hbar c}\left[\mathscr{K}, Q\right]\Phi \Big\} dv; \end{aligned}\right\} \tag{83}$$

it is readily seen that the factors of the products occurring in $\mathscr{W}_n$—and consequently also of those occurring in w—are commutable, so that both $\mathscr{W}_n$ and w are real operators. Using the formulae (9), (14), (30), the term (82) of direct coupling between the nuclear particles is easily brought into the form

$$\left.\begin{aligned} \mathscr{W}_n = \int \left\{ \vec{M}(\vec{x}') \wedge \vec{S}(\vec{x}) + N(\vec{x}')\,\vec{T}(\vec{x}) - R(\vec{x}')\,\vec{P}(\vec{x}) \right\} \\ \cdot \operatorname{grad} \varphi \, dv\, dv', \end{aligned}\right\} \tag{84}$$

or, with the expressions (3), (4), (23), (24), and omitting the undefined contributions which correspond to self-energies ($i = k$),

$$\left.\begin{aligned} \mathscr{W}_n &= \frac{1}{2}\sum_{i \neq k}{}' \left(\tau^{(i)}\tau^{(k)}\right)\left(\vec{\chi}^{(ik)}\operatorname{grad}^{(k)} + \vec{\chi}^{(ki)}\operatorname{grad}^{(i)}\right)\varphi\left(r^{(ik)}\right), \\ &\text{with} \\ \vec{\chi}^{(ik)} &= \frac{g_1 g_2}{\kappa}\left(\rho_1^{(i)}\rho_3^{(k)}\vec{\sigma}^{(i)} \wedge \vec{\sigma}^{(k)} + \rho_2^{(i)}\vec{\sigma}^{(i)}\right) - \frac{f_1 f_2}{\kappa}\rho_2^{(i)}\vec{\sigma}^{(k)}. \end{aligned}\right\} \tag{85}$$

Summing up, we find, on account of (76), (78), and (81), for the expression (69) of the Hamiltonian as a function of the new variables,

$$
\left.\begin{aligned}
\mathcal{H}_{(\text{new var.})} &= \mathcal{H}_n + \mathcal{H}_f \\
&- \overset{\odot}{\mathcal{K}} + \frac{1}{2}\int\left\{-\kappa^{-2}\boldsymbol{n}\,\mathrm{div}\,\vec{\boldsymbol{F}} + \vec{\boldsymbol{s}}\,\mathrm{rot}\,\vec{\boldsymbol{U}} - \vec{\boldsymbol{p}}\,\mathrm{grad}\,\boldsymbol{\Psi}\right\} dv + \text{conj.} \\
&- \sum_{l=1}^{\infty} \frac{1}{(l+1)!}\left\{\frac{i}{\hbar c}\mathcal{K}, \overset{\odot}{\mathcal{K}}\right\}^l \\
&\qquad - \frac{1}{2}\int dv\,dv'\left(\kappa^{-2}\boldsymbol{n}(\vec{x})\,\boldsymbol{n}(\vec{x}') + \vec{\boldsymbol{s}}(\vec{x})\,\vec{\boldsymbol{s}}(\vec{x}')\right)\Delta\varphi \\
&+ \mathcal{W}_n + \mathcal{W} + w + \sum_{l=1}^{\infty}\frac{1}{(l+1)!}\left\{\frac{i}{\hbar c}\mathcal{K}, \mathcal{W}_n + w\right\}^l.
\end{aligned}\right\} \quad (86)
$$

It consists of the Hamiltonian $\mathcal{H}_n$, given by (64), (65), of the system of nuclear particles with static interactions, the Hamiltonian $\mathcal{H}_f$, defined by (66), of the pure meson fields, and several coupling terms, depending on various quantities defined by (67), (74), (80), (83) and (85); it should be remembered that the expression (86) is exact only to the first order in the velocities of the heavy particles. Further, we shall, in all terms containing $\overset{\odot}{\mathcal{K}}$, disregard the contributions arising from the mass-terms in the kinetic energy (which represent couplings between the heavy particles and the meson fields), since they involve the factor $\frac{(M_N - M_P)\,c}{\hbar\kappa}$, which is small compared with $\frac{G}{4\pi\hbar c}$.

For later purposes, let us write down the field equations derived from (86), when we look apart from all velocity dependence, and thus in particular cancel the terms of the last line in (86):

$$
\left.\begin{aligned}
&\left\{\begin{aligned}
\dot{\vec{\boldsymbol{U}}} &= -\vec{\boldsymbol{F}} + \kappa^{-2}\,\mathrm{grad}\,\mathrm{div}\,\vec{\boldsymbol{F}} - \overset{\odot}{\vec{\boldsymbol{U}}}{}^{\circ} + \vec{\boldsymbol{\Theta}} \\
\dot{\vec{\boldsymbol{F}}} &= \kappa^2\vec{\boldsymbol{U}} + \mathrm{rot}\,\mathrm{rot}\,\vec{\boldsymbol{U}} - \overset{\odot}{\vec{\boldsymbol{F}}}{}^{\circ} + \vec{\boldsymbol{\Theta}},
\end{aligned}\right. \\
&\left\{\begin{aligned}
\dot{\boldsymbol{\Psi}} &= \boldsymbol{\Phi} - \overset{\odot}{\boldsymbol{\Psi}}{}^{\circ} + \boldsymbol{\Theta} \\
-\dot{\boldsymbol{\Phi}} &= \kappa^2\boldsymbol{\Psi} - \Delta\boldsymbol{\Psi} + \boldsymbol{\Theta};
\end{aligned}\right.
\end{aligned}\right\} \quad (87)
$$

here the "stationary" source densities $-\overset{\odot}{\vec{\boldsymbol{U}}}{}^{\circ}$, $-\overset{\odot}{\vec{\boldsymbol{F}}}{}^{\circ}$, $-\overset{\odot}{\boldsymbol{\Psi}}$ arise from the term $-\overset{\odot}{\mathcal{K}}$ in (86), while the symbols $\vec{\boldsymbol{\Theta}}$, $\boldsymbol{\Theta}$ denote the source densities derived from all other velocity-independent coupling terms in the Hamiltonian.

In the expressions of the source-densities which we have called "stationary", the velocity-dependent terms are of course to be cancelled, so that these sources just correspond to the precessions of the spins and isotopic spins of the heavy particles under the influence of the static forces. While they are of course independent of the field variables, the other source densities $\vec{\boldsymbol{\Theta}}$, $\boldsymbol{\Theta}$ do not contain any field-independent part, since the terms in (86) from which they are derived are at least quadratic functions of the field variables. This follows at once from the remark that $\left\{\frac{i}{\hbar c}\mathcal{K}, A\right\}^{l}$ $(l \geq 1)$, where A is any function of the variables of the heavy particles alone, is homogeneous of order l in the field variables. Such a property would be quite trivial, were it not for the non-commutability of the field components $\vec{\boldsymbol{U}}$ and $\vec{\boldsymbol{F}}$; since these, however, occur only in the combinations $\int \vec{\boldsymbol{F}}^{\circ}\vec{\boldsymbol{U}}\,dv$ and $\int \vec{\boldsymbol{U}}^{\circ}\vec{\boldsymbol{F}}\,dv$, a reduction of the order of $\left\{\frac{i}{\hbar c}\mathcal{K}, A\right\}^{l}$ in the field variables could only arise through a factor

$$\frac{i}{\hbar c}\int \vec{\boldsymbol{F}}^{\circ}(\vec{x})\cdot\left[\vec{\boldsymbol{U}}(\vec{x}),\ \vec{\boldsymbol{F}}(\vec{x}')\right]\cdot\vec{\boldsymbol{U}}^{\circ}(\vec{x}')\,dv\,dv'$$
$$=\int \vec{\boldsymbol{F}}^{\circ}(\vec{x})\,\vec{\boldsymbol{U}}^{\circ}(\vec{x})\,dv,$$

which, according to (9) and $\operatorname{div}\vec{\boldsymbol{U}}^{\circ}=0$, reduces to zero.

We are now prepared to discuss, in the next section, to what extent an unambiguous solution of the field equations is possible, and whether the non-static meson fields so obtained are of any importance for the properties of the stationary states of nuclear systems.

5. Physical interpretation of the formalism.

The convergence difficulties which prevent a consistent combination of the field concept with the ideas of quantum theory, oblige us to restrict in a suitable way the use of the formalism developed in the preceding sections. In the case of electrodynamics, the choice of the required restrictions is guided by the well-known correspondence argument[11]. It is true that we have in the present case, on account of the large meson mass, no empirical evidence of field properties of mesons in a domain where quantum effects would be negligible; but just in the critical region, defined by (51), with which we are concerned in the problem of nuclear fields, the influence of the meson mass on the properties of the field becomes unimportant. It therefore seems natural to adopt, in discussing the limitations of the formalism of meson theory, a point of view closely analogous to that of quantum electrodynamics.

The canonical equations derived from the Hamiltonian should thus not be considered as an exact system of equations, but solved by a process of successive approximations in which, starting from a suitably defined unperturbed system, the calculation of the solution corresponding to a given initial state of this system should not be carried further than the first step leading to a non-vanishing result for the effect under consideration; and such results should of course only be considered as reliable if even this first step does not involve any ambiguity.

The justification of such a procedure, as well as the precise way in which it is to be conducted, can only be derived from the treatment of the "corresponding" problem, in which the meson fields are not quantized. If, in such a treatment, we start from a state in which all the field components are zero everywhere, it is clear that, provided the procedure converges, the various interactions between heavy particles and meson fields, and the resulting source densities in the meson field equations, are to be regarded as perturbations of increasing order according to the power to which they contain the field components. We have further to demand that, in the application of the method of successive approximations thus defined to the system of unquantized meson fields, the effects of higher order than those which should alone be retained according to the above prescription be actually negligible.

Looking from this extended "correspondence" point of view at the Hamiltonian (56) expressed by the original variables, we see that a strict application of the prescription just formulated would not lead to any reliable estimate of the binding energy of a nuclear system: it is true that we could in this way derive the expression of a direct coupling between the heavy particles, but we would not be justified in treating such an expression as an operator which, together with the kinetic energy, would determine the stationary states of the nuclear system. A quite analogous situation would of course be met with in electrodynamics, if the same prescription were applied to the Hamiltonian including the longitudinal part of the electric field and vector potential; in fact, a true correspondence with classical theory is only achieved when these longitudinal fields have been eliminated and replaced by the static

Coulomb interaction. It is just the purpose of the canonical transformation discussed in Part I to obtain for the treatment of nuclear systems a starting point comparable to the quantum mechanics of atomic systems. Although the seperation of the static part of the fields, performed in this manner, is not a relativistically invariant operation, we have in either case a natural frame of reference (viz. that in which the centre of gravity of the system is at rest), with respect to which such a separation has a well-defined meaning.

We have thus to examine to what extent a consistent use of the scheme based on the Hamiltonian (86), expressed in terms of the transformed variables, may be found by means of the "correspondence" prescription formulated above. For this purpose, we shall first discuss the convergence of the corresponding theory in which the meson fields are not quantized, and afterwards the limitations imposed on the theory by the difficulties of field quantization. In this discussion, we shall of course be concerned with two distinct problems, viz. the influence of the non-static forces on the stationary states of nuclear systems, and the transition processes due to the interaction between such systems and the meson fields.

As regards the calculation of the non-static interaction between heavy particles, arising from unquantized meson fields, we shall first investigate the non-static meson fields due to the stationary source densities in (87). Since we are interested in the values of these fields in the region occupied by the nuclear system, we may neglect the retardation effects; remembering that div $\vec{\boldsymbol{U}}^{\circ} = 0$, rot $\vec{\boldsymbol{F}}^{\circ} = 0$, we therefore get immediately from (87), for the quasi-stationary fields due to the spin precessions,

$$\vec{U}_s = \kappa^{-2} \overset{\odot}{\vec{F}}{}^{\circ}, \quad \vec{F}_s = -\overset{\odot}{\vec{U}}{}^{\circ}, \quad \mathbf{\Phi}_s = \overset{\odot}{\mathbf{\Psi}}{}^{\circ}, \quad \mathbf{\Psi}_s = 0. \tag{88}$$

In order to estimate the influence of such fields on the stationary states of nuclear systems, we shall compare the interaction energy $\mathcal{V}_s$ to which they give rise with the static interaction $\mathcal{V}_n$. A sufficiently accurate expression of $\mathcal{V}_s$ is obtained by inserting the fields (88) in the corresponding approximate field Hamiltonian $\mathcal{H}_f - \overset{\odot}{\mathcal{K}}$. Using (13) and

$$\int \varphi\left(\left|\vec{x}-\vec{x}'\right|\right) \varphi\left(\left|\vec{x}-\vec{x}''\right|\right) dv = \varphi\left(\left|\vec{x}'-\vec{x}''\right|\right) \cdot \frac{\left|\vec{x}'-\vec{x}''\right|}{2\kappa}, \tag{89}$$

we get

$$\left.\begin{aligned} \mathcal{V}_s = -\frac{1}{2} \int dv'\, dv'' \left\{ \kappa^{-2} \overset{\odot}{\mathbf{N}}(\vec{x}')\, \overset{\odot}{\mathbf{N}}(\vec{x}'') + \overset{\odot}{\vec{S}}(\vec{x}')\, \overset{\odot}{\vec{S}}(\vec{x}'') \right\} \\ \varphi\left(\left|\vec{x}'-\vec{x}''\right|\right) \left(1 - \frac{\kappa\left|\vec{x}'-\vec{x}''\right|}{2}\right); \end{aligned}\right\} \tag{90}$$

there occurs a further term

$$\left.\begin{aligned} -\frac{1}{2} \int dv\, dv'\, dv'' \Big\{ \left(\overset{\odot}{\vec{S}}(\vec{x}')\, \mathrm{grad}\, \varphi\left(\left|\vec{x}-\vec{x}'\right|\right)\right) \left(\overset{\odot}{\vec{S}}(\vec{x}'')\, \mathrm{grad}\, \varphi\left(\left|\vec{x}-\vec{x}''\right|\right)\right) \\ - \left(\overset{\odot}{\vec{P}}(\vec{x}')\, \mathrm{grad}\, \varphi\left(\left|\vec{x}-\vec{x}'\right|\right)\right) \left(\overset{\odot}{\vec{P}}(\vec{x}'')\, \mathrm{grad}\, \varphi\left(\left|\vec{x}-\vec{x}''\right|\right)\right) \Big\} \end{aligned}\right\} \tag{9}$$

which vanishes to the first order in the velocities. For estimates of order of magnitude, it will be sufficient to consider a pair of nuclear particles at some fixed distance $r\,(<\kappa^{-1})$, the different powers of this distance representing the order of magnitude of the expectation values of the corresponding quantities in the stationary state concerned, provided these expectation values are finite. The relative orders of magnitude of velocity-independent terms of interaction are then conveniently expressed in powers of the parameter

$$\gamma = \frac{G}{4\pi\hbar c}\cdot\frac{1}{\kappa r} \quad (G \backsim g_1^2 \text{ or } g_2^2). \tag{92}$$

Thus, by reference to the formulae (49) (with $G_3 = 0$), we see that

$$\mathcal{V}_s \backsim \gamma^2 \, \mathcal{V}_n; \tag{93}$$

the requirement that $\mathcal{V}_s$ be small compared to $\mathcal{V}_n$ leads therefore precisely to the introduction of the critical distance r_c defined by (51). Passing to the higher approximations in (87), and observing that the order of magnitude of the quantity $\mathcal{K}$, in which the stationary fields (88) have been introduced, is just γ^2, we may easily verify that all successive contributions to the interaction between nuclear particles differ as to order of magnitude at most by powers of γ. If we now also take into account the velocity-dependent contributions, we have to introduce, besides γ, another parameter

$$\beta = \frac{v}{c}\cdot\frac{1}{\kappa r}, \tag{94}$$

where v represents the order of magnitude of the velocities of the heavy particles. The main velocity-dependent contribution to the coupling between nuclear particles is the term

$$\mathcal{W}_n \backsim \beta \, \mathcal{V}_n \tag{95}$$

which, since it does not involve the meson fields, may from our present point of view simply be included in the Hamiltonian of the unperturbed system of nuclear particles, where it will be considered as a correction to the static interaction. The other velocity-dependent couplings, which all represent interactions between the heavy particles and the meson fields, will be seen to give contributions of higher order in β or γ. This holds further for the contributions arising from the terms of second order in the velocities, which we have omitted from the Hamiltonian (86). Since, for actual nuclear systems, β is numerically about the same as γ, the convergence condition $\beta \ll 1$ leads again practically to the value of the critical distance r_c given by (51). It should be observed that, if we had performed the preceding discussion in the case of the pure vector meson theory or any other including static couplings of the dipole interaction type, we would have had to take account of velocity-independent terms corresponding to the first terms of (77) and (91), and we would have been led to a condition involving the critical distance (50) instead of (51).

As is well-known, the quantization of the meson fields implies the occurrence of fluctuating fields even in the absence of any nuclear matter, and the interaction of such zero-fields with any nuclear particle will give rise to an infinite contribution to the self-energy of the particle. While, as we have just seen, the interaction between nuclear particles due to unquantized meson fields could in principle be calculated to any approximation, provided only that the distances involved are larger than the critical distance r_c, the necessity of avoiding the infinite self-energies due to the zero-fields forces us, in accordance with our general prescription, to discard entirely all non-static terms of direct coupling between nuclear particles (except of course the term $\mathcal{W}_n$, included in the unperturbed Hamiltonian of the system of nuclear particles).

The consideration of the probabilities of transition processes due to the interaction between heavy particles and meson fields imposes on the theory, according to Heisenberg[12)], a radical limitation arising from the increase of the probabilities of "explosive" processes, when the energy involved becomes large compared with the rest-energy of the mesons. In fact, if λ is the wave-length of the mesons taking part in the process considered, it is easily seen that, for the first explosive processes to set in when the energy increases, the transition probabilities are proportional to some power of the parameter

$$\alpha = \frac{G}{4\pi\hbar c}\frac{1}{(\kappa\lambda)^2}. \tag{96}$$

The order of magnitude of the energies for which such explosions set in is thus connected, according to (96), with a critical length given by

$$\kappa r_0 = \sqrt{\frac{G}{4\pi\hbar c}}, \tag{97}$$

which is smaller than the critical distance r_c in the pure vector meson theory, given by (50), but larger than the distance (51) corresponding to the form of the theory which we have adopted. This limitation affects equally any form of meson theory, except[19] a theory of purely neutral meson fields involving only couplings which depend on the fundamental constant g_1.

If, as advocated by HEISENBERG, the critical length r_0 has a universal significance, in the sense that the usual concepts of field theory would not be applicable within regions of a linear extension smaller than r_0, we have to expect in our case a somewhat more rigorous restriction of the domain of applicability of the interaction potentials $\mathfrak{V}_n$ and $\mathfrak{W}_n$ than that defined by the critical distance r_c. Still, there remains a range of distances between r_0 and κ^{-1}, where the form of these potentials is significant and where

4*

the treatment of the stationary states of nuclear systems outlined above yields (in contrast to the case of vector meson theory) unambiguous results*. It seems probable —though by no means certain[20]— that such results would not be essentially affected by the modifications which a rational introduction of the universal length r_0 in the theory would involve, since these modifications would presumably be confined mainly to regions of linear extensions smaller than r_0, which are of minor importance for the determination of stationary states.

From the preceding discussion we conclude that, if we treat the Hamiltonian (86) from the correspondence point of view described in this section, we obtain as the only significant interactions between nuclear particles those defined by the potentials $\mathfrak{V}_n$ and $\mathfrak{W}_n$; the other terms of coupling between nuclear particles and meson fields may be used only to calculate, in conformity with the correspondence prescription, the probabilities of the various transition processes involving energies of the mesons not large compared with their rest-energy.

As regards the determination of such transition probabilities, it should be observed that, for purposes of practical calculations, it would in most cases be more advantageous to apply the procedure of successive approximations described to the Hamiltonian (56) expressed in terms of the old variables, since the operator of interaction between nuclear particles and meson fields involved in this Hamiltonian has a much simpler form. If the use of the original

* It will be noticed that the existence of the universal length r_0 would deprive of any well-defined meaning all potentials of direct interaction between three or more nuclear particles, which, as shown by the preceding discussion, become important only for distances of the order r_c. Cf. [19a]).

variables is thus adopted,—as has naturally been the case in all calculations hitherto carried out,—it is obviously permissible to add to the Hamiltonian of the unperturbed nuclear system the interaction potential $\mathcal{V}_n + \mathcal{W}_n$, since this operator does not give any contribution to the matrix elements contained in the expression of the transition probabilities. From the character of the processes involved, it is clear that results obtained in this way should be entirely equivalent to those of calculations using the transformed variables, in spite of the widely different forms of the interaction operator in the two cases. In fact, the probabilities of such processes are proportional to the square of the matrix elements of the operator $e^{-\frac{i\mathcal{H}t}{\hbar}}$ for initial and final states of the whole system, consisting of some stationary state of the unperturbed nuclear system and (at least for one of the two states) one or more meson wave-packets at large distances from the nucleus. If we use the new variables, and if we also apply the transformation $\mathcal{S}$ to the scheme of representation, we should, according to the conclusions of Section 3, take as wave-functions describing the initial and final states the same functions in the new representation as in the old. Strictly speaking, we have therefore to do with different states in the two cases, but the difference is vanishingly small for the kind of states concerned, since the transformation $\mathcal{S}$ modifies only the form of the meson field components in the neighbourhood of the nuclear particles.

As an illustration of this point, let us consider the interaction between meson fields and a single nuclear particle; the interaction operators occurring in the two forms of the Hamiltonian are, according to (56) and (86),

$$\left.\begin{aligned}\Theta_{(\text{old var.})} &= \Theta_0 + \mathcal{U}\\ \Omega_{(\text{new var.})} &= \frac{i}{\hbar c}\left[\mathcal{K},\ \mathcal{H}_k\right] + \mathcal{U} + \begin{cases}\text{terms of higher order}\\ \text{in field components,}\end{cases}\end{aligned}\right\} \quad (98)$$

where $\mathcal{U}$ is given by (67) and

$$\Theta_0 = \int \left\{ -\kappa^{-2}\boldsymbol{N}\,\mathrm{div}\,\vec{\boldsymbol{F}} + \vec{\boldsymbol{S}}\,\mathrm{rot}\,\vec{\boldsymbol{U}} - \vec{\boldsymbol{P}}\,\mathrm{grad}\,\boldsymbol{\Psi}\right\} dv. \quad (99)$$

A striking difference between these operators is that all velocity-independent terms of first order in the field components have disappeared from the operator corresponding to the use of the new variables. This indicates that the velocity-independent interaction Θ_0 in the old variables actually gives only velocity-dependent contributions to the probabilities of emission or absorption of single mesons by a nuclear particle. In this simple case, the equivalence of the two modes of calculation of these probabilities is readily verified as follows. Observing that $\Theta_0 = -\frac{i}{\hbar c}\left[\mathcal{K},\ \mathcal{H}_f\right]$, we may write

$$\Omega = \Theta + \frac{i}{\hbar c}\left[\mathcal{K},\ \mathcal{H}_k + \mathcal{H}_f\right] + \begin{Bmatrix}\text{terms of higher}\\ \text{order.}\end{Bmatrix} \quad (100)$$

The probability per unit time of a process of emission or absorption of a single meson is in first approximation proportional to the square of the same matrix element of either Θ or Ω, corresponding to two states of the same unperturbed energy $\mathcal{H}_k + \mathcal{H}_f$. Now, according to (100), such matrix elements are actually equal to the approximation considered, since the corresponding matrix element of

$$\frac{i}{\hbar c}\left\{\mathcal{K}\left(\mathcal{H}_k + \mathcal{H}_f\right) - \left(\mathcal{H}_k + \mathcal{H}_f\right)\mathcal{K}\right\}$$

is zero.

We should finally like to point out that the use of the transformed variables brings considerable simplification in the discussion of the processes due to the interaction of meson fields and nuclear particles with electromagnetic fields or with electrons and neutrinos. To such problems, which include "optical" properties of nuclei and β-disintegration, we shall come back in later papers.

Part III.

Stationary states of the deuteron.

We shall now apply the potential of interaction between nuclear particles $\mathcal{V}_n + \mathcal{W}_n$, derived from the special form of meson theory proposed in this paper, to the study of the stationary states of the simplest nuclear system, the deuteron. In this discussion, the velocity-dependent coupling will be treated as a perturbation. We therefore begin by recalling the main features of the solution of the problem for a potential of the form $\mathcal{V}_n$, as given by Kemmer[18], and derive from it a rough fixation of the numerical values of the constants $|g_1|$ and $|g_2|$. We then estimate the influence of the perturbation potential $\mathcal{W}_n$ on the binding energy and eigenfunction of the ground state and, finally, apply the last result to the calculation of the electric quadrupole moment of the deuteron in this state.

1. Stationary states of the deuteron as determined by the static potential.

Let us first consider the stationary states of the deuteron as determined by the static potential $\mathcal{V}_n$. Following Kemmer[18], we describe these states, in the frame of reference in which the centre of gravity is at rest, by the proper solutions of the equation

$$\left\{\frac{\hbar c}{i}\vec{\alpha}\,\mathrm{grad}+\beta Mc^2+\mathcal{V}_n(r)\right\}\Psi_E(\vec{x})=E\Psi_E(\vec{x}), \quad (101)$$

where $\vec{x}=\vec{x}^N-\vec{x}^P$ are the relative coordinates*, $r=|\vec{x}|$, $\vec{\alpha}=\vec{\alpha}^N-\vec{\alpha}^P$, $\beta=\vec{\rho}_3^N+\vec{\rho}_3^P$ and $M\backsim M_N\backsim M_P$. As shown by KEMMER, the non-trivial proper solutions reduce to three types, which he denotes by I *a*, I *b* and II *b*, *a* and *b* referring to the even or odd character of the eigenfunctions. In the non-relativistic approximation, types I and II correspond respectively to the triplet and to the singlet system; in this approximation, each state is characterized not only by the energy E and total angular momentum j, but further by the orbital momentum l, and we have

$$\begin{cases}\text{for type I}\,a: & l=j\pm1,\\ \text{for types I}\,b\text{ and II}\,b: & l=j.\end{cases}$$

The radial part of the "large" (i. e. velocity-independent) components of the proper solutions is in all cases determined by a SCHRÖDINGER equation

$$\left\{\frac{\hbar^2}{M}\left(\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right)+\Gamma\frac{e^{-\kappa r}}{r}+E\right\}R_l(r)=0, \quad (102)$$

where

$$\left.\begin{aligned}&\text{for type I}\,a: && \Gamma=\left[1-2(-1)^j\right]\frac{(g_1^2+g_2^2)}{4\pi},\\ &\text{for type I}\,b: && \Gamma=\left[1-2(-1)^{j+1}\right]\frac{g_1^2+g_2^2}{4\pi},\\ &\text{for type II}\,b: && \Gamma=\left[2(-1)^j-1\right]\frac{-g_1^2+3g_2^2}{4\pi}.\end{aligned}\right\} \quad (103)$$

In the present paper, we shall confine ourselves to a provisory survey. A more exact treatment of the equation

* Quantities referring to the two particles of the deuteron are distinguished by the upper indices N and P.

(102) is being carried out by Dr. L. Hulthén, who will also apply his results to a discussion of the problems here treated.

As regards the description of the ground state and of the excited ${}^1S$ state revealed by scattering experiments, we may, if we assume that the proper energy of the latter is approximately zero, use the available results of numerical integrations[21)] of the equation (102) for S states and $E \leq 0$. These results may to a fair approximation be summarized[22)] in the formula

$$\frac{\Gamma}{\hbar c} = 1.69 \frac{M_m}{M} + \sqrt{5 \frac{|E|}{Mc^2}}, \tag{104}$$

where M_m represents the mass of the meson. For the two S-states concerned, which are of types Ia and IIb with $j = 1$ and $j = 0$ respectively, we therefore get

$$\left.\begin{aligned} \frac{3(g_1^2 + g_2^2)}{4\pi \hbar c} &= 1.69 \frac{M_m}{M} + \sqrt{5 \frac{|E_0|}{Mc^2}}, \\ \frac{-g_1^2 + 3g_2^2}{4\pi \hbar c} &= 1.69 \frac{M_m}{M}, \end{aligned}\right\} \tag{105}$$

$|E_0| \backsim 0.0023\, Mc^2$ denoting the binding energy of the ground state. From (105) we find

$$\frac{g_1^2}{4\pi \hbar c} = \frac{1}{4} \sqrt{5 \frac{|E_0|}{Mc^2}} = 0.027, \tag{106}$$

—independent (to the approximation used) of the value of the meson mass,—and further

$$\frac{g_2^2}{4\pi \hbar c} = 0.56 \frac{M_m}{M} + 0.009, \tag{107}$$

showing that $|g_2|$ is essentially determined by the value of the meson mass only. We get

$$\left.\begin{aligned} &\text{for } \frac{M_m}{M} = \frac{1}{10}: \quad \frac{g_2^2}{4\pi\hbar c} = 0.065\,, \\ &\text{for } \frac{M_m}{M} = \frac{1}{20}: \quad \frac{g_2^2}{4\pi\hbar c} = 0.037\,. \end{aligned}\right\} \qquad (108)$$

The numerical values (106) and (108) provide the justification of the general statement on p. 32, that the mean distance between any pair of particles in stationary states of nuclei is large compared with the critical distance r_c defined by (51). In fact, such mean distances will of course be at least of the order of magnitude κ^{-1}. This may in particular be seen for the ground state of the deuteron by using for the radial wave-function the approximate analytical representation given by WILSON[21]:

$$\left.\begin{aligned} &R_0(r) = \sqrt{\frac{\alpha^3 \kappa^3}{2}}\, e^{-\frac{\alpha\kappa r}{2}} \cdot r\,, \\ &\text{with} \begin{cases} \alpha = 2.13, & \text{for } \dfrac{M_m}{M} = \dfrac{1}{10}, \\ \alpha = 3.3, & \text{for } \dfrac{M_m}{M} = \dfrac{1}{20}. \end{cases} \end{aligned}\right\} \qquad (109)$$

The 16 components of the eigenfunction of any stationary state of (101), characterized by the eigenvalues of ρ_3^P, σ_3^P; ρ_3^N, σ_3^N, may to the first order in the velocities be written, with reference to the table on p. 52 and formulae (6) to (14) in KEMMER's paper, in the form

$$\Psi = \overset{(0)}{\Psi} + \overset{(1)}{\Psi}, \qquad (110)$$

where the velocity-independent term

$$\overset{(0)}{\Psi} = \delta_{\rho_3^P, 1}\, \delta_{\rho_3^N, 1}\, Z_{\sigma_3^P \sigma_3^N} \tag{111}$$

and the first order term

$$\overset{(1)}{\Psi} = \delta_{\rho_3^P, 1}\, \delta_{\rho_3^N, -1}\, z_{\sigma_3^P \sigma_3^N} \mp \delta_{\rho_3^P, -1}\, \delta_{\rho_3^N, 1}\, z_{\sigma_3^N \sigma_3^P}, \tag{112}$$

the upper sign corresponding to type I, the lower to type II; Z is symmetrical with respect to σ_3^P, σ_3^N for type I, antisymmetrical for type II. We shall, in the following, only use the explicit expression of the Z and z for states of type Ia and $j = 1$ given by the formulae (113) and (114) on p. 61. In these formulae, $Y_l^{(m)}$ are the normalized Legendre functions

$$\left.\begin{aligned} Y_l^{(m)} &= \frac{1}{\sqrt{2\pi}} e^{im\varphi} P_{l,m}(\cos\theta), \\ P_{l,m}(x) &= \sqrt{\frac{2l+1}{2} \cdot \frac{(l-m)!}{(l+m)!}} \frac{1}{2^l \cdot l!} (1-x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l, \end{aligned}\right\} \tag{115}$$

and the numerical factors have been chosen so as to normalize the total eigenfunction to unity, provided the radial factor R_l is normalized in the usual way.

$\underline{l = 0:}$

$$\left\| \overset{(0)}{Z}{}^{P\,N}_{\sigma_3\sigma_3} \right\| = \left\| \begin{matrix} \sqrt{m(m+1)}\,Y_0^{(m-1)} & -\sqrt{(m+1)(1-m)}\,Y_0^{(m)} \\ -\sqrt{(m+1)(1-m)}\,Y_0^{(m)} & \sqrt{m(m-1)}\,Y_0^{(m+1)} \end{matrix} \right\| \cdot \frac{1}{\sqrt{2}}\,\frac{R_0(r)}{r},$$

$$\left\| \overset{(0)}{z}{}^{P\,N}_{\sigma_3\sigma_3} \right\| = \left\| \begin{matrix} -\sqrt{(1+m)(2-m)}\,Y_1^{(m-1)} & -(m+1)\,Y_1^{(m)} \\ -(m-1)\,Y_1^{(m)} & \sqrt{(1-m)(2+m)}\,Y_1^{(m+1)} \end{matrix} \right\| \cdot \frac{1}{2\sqrt{2}}\,\frac{C_0(r)}{r}, \tag{113}$$

$$C_0 = \frac{\hbar}{Mc}\cdot\frac{i}{\sqrt{3}}\left(\frac{d}{dr}-\frac{1}{r}\right)R_0;$$

$\underline{l = 2:}$

$$\left\| \overset{(2)}{Z}{}^{P\,N}_{\sigma_3\sigma_3} \right\| = \left\| \begin{matrix} \sqrt{(2-m)(3-m)}\,Y_2^{(m-1)} & \sqrt{(2+m)(2-m)}\,Y_2^{(m)} \\ \sqrt{(2+m)(2-m)}\,Y_2^{(m)} & \sqrt{(2+m)(3+m)}\,Y_2^{(m+1)} \end{matrix} \right\| \cdot \frac{1}{2\sqrt{5}}\,\frac{R_2(r)}{r},$$

$$\left\| \overset{(2)}{z}{}^{P\,N}_{\sigma_3\sigma_3} \right\| = \left\| \begin{matrix} -\sqrt{(1+m)(2-m)}\,Y_1^{(m-1)} & -(m-2)\,Y_1^{(m)} \\ -(m+2)\,Y_1^{(m)} & \sqrt{(1-m)(2+m)}\,Y_1^{(m+1)} \end{matrix} \right\| \cdot \frac{1}{4}\,\frac{C_2(r)}{r}, \tag{114}$$

$$C_2 = \frac{\hbar}{Mc}\cdot\frac{i}{\sqrt{3}}\left(\frac{d}{dr}+\frac{2}{r}\right)R_2.$$

2. First order perturbation of the ground state by the non-static potential.

The non-static potential may, according to (85), be written as

$$\left.\begin{aligned} \mathcal{W}_n &= \left(\overset{N}{\boldsymbol{\tau}}\overset{P}{\boldsymbol{\tau}}\right)\left(\overset{\rightarrow NP}{\chi} - \overset{\rightarrow PN}{\chi}\right)\operatorname{grad}\varphi, \\ \overset{\rightarrow NP}{\chi} &= \frac{f_1 f_2}{\varkappa}\overset{\rightarrow P}{\sigma}\overset{N}{\rho_2} - \frac{g_1 g_2}{\varkappa}\left(\overset{\rightarrow N}{\sigma} + i\overset{\rightarrow N}{\sigma}\wedge\overset{\rightarrow P}{\sigma}\overset{N}{\rho_3}\overset{P}{\rho_3}\right)\overset{N}{\rho_2}. \end{aligned}\right\} \quad (116)$$

Since this operator is invariant for rotations and for reflections with respect to the origin of the relative space coordinates, the matrix element

$$(B|\mathcal{W}_n|A) = \int \Psi_B^* \, \mathcal{W}_n \Psi_A \, dv$$

is $\neq 0$ only if the states A and B have the same quantum numbers j and m and the same even or odd character a or b. Taking account of the symmetry properties of $\overset{(0)}{\Psi}$ and $\overset{(1)}{\Psi}$ with respect to the spin coordinates ρ_3 and σ_3 of neutron and proton, it is easily seen that, to the first order in the velocities, we then have

$$(B|\mathcal{W}_n|A) = \int \overset{(0)}{\Psi_B^*}\mathcal{W}_n \overset{(1)}{\Psi_A}\, dv + \int \overset{(1)}{\Psi_B^*}\mathcal{W}_n \overset{(0)}{\Psi_A}\, dv, \quad (117)$$

if the states A and B belong to the same type I or II, while no intercombinations between states of types I b and II b occur. Since, for a given type, $\overset{(0)}{\Psi}$ and $\overset{(1)}{\Psi}$ are of different symmetry with respect to the ρ_3's and σ_3's, we have

$$\overset{(0)}{\Psi_B^*}\overset{\rightarrow NP}{\chi}\overset{(1)}{\Psi_A} = -\overset{(0)}{\Psi_B^*}\overset{\rightarrow PN}{\chi}\overset{(1)}{\Psi_A};$$

observing further that $\overset{(0)}{\Psi}$ corresponds to the eigenvalue 1 of $\overset{N}{\rho_3}\overset{P}{\rho_3}$, and that

$$\vec{\sigma}^N + i\vec{\sigma}^N \wedge \vec{\sigma}^P = \vec{\sigma}^P \left(\vec{\sigma}^N \vec{\sigma}^P\right),$$

we get

$$\int \overset{(0)}{\Psi}{}^{\star}_B \mathcal{U}_n \overset{(1)}{\Psi}_A \, dv$$

$$= 2 \int \overset{(0)}{\Psi}{}^{\star}_B \left(\mathbf{\tau}^N \mathbf{\tau}^P\right) \overset{N \to P}{\rho_2 \, \sigma} \operatorname{grad} \varphi \left\{ \frac{f_1 f_2}{\kappa} - \frac{g_1 g_2}{\kappa} \left(\vec{\sigma}^N \vec{\sigma}^P\right) \right\} \overset{(1)}{\Psi}_A \, dv$$

or, using the expressions (111), (112),

$$\left.\begin{aligned} &\int \overset{(0)}{\Psi}{}^{\star}_B \mathcal{U}_n \overset{(1)}{\Psi}_A \, dv \\ &= -2i \int Z^{\star}_B \left(\mathbf{\tau}^N \mathbf{\tau}^P\right) \vec{\sigma}^P \operatorname{grad} \varphi \left\{ \frac{f_1 f_2}{\kappa} - \frac{g_1 g_2}{\kappa} \left(\vec{\sigma}^N \vec{\sigma}^P\right) \right\} z_A \, dv. \end{aligned}\right\} \quad (118)$$

Taking now as state A the ground state and as state B any other state combining with it, i. e. a state of type I a and energy E, with $j = 1$, and $l = 0$ or 2, we may easily from (117), (118) calculate the corresponding matrix elements with the help of the representation (113), (114); since these matrix elements must obviously be independent of m, it is only necessary to carry out the calculation for an arbitrarily chosen value of m. The result is

$$\left.\begin{aligned} &(E, l = 0 \,|\, \mathcal{U}_n \,|\, 0) \\ &= -\frac{5 g_1 g_2 - f_1 f_2}{4 \pi \hbar c} \left(\frac{M_m}{M}\right)^2 \cdot M c^2 \cdot \frac{4\pi}{\kappa^3} \int_0^\infty \frac{d\varphi}{dr} \left(\frac{d}{dr} - \frac{2}{r}\right) \left(R_0 \cdot R_0^{(E)\star}\right) dr, \\ &(E, l = 2 \,|\, \mathcal{U}_n \,|\, 0) \\ &= \frac{2\sqrt{2}\,(g_1 g_2 + f_1 f_2)}{4 \pi \hbar c} \cdot \left(\frac{M_m}{M}\right)^2 M c^2 \cdot \frac{4\pi}{\kappa^3} \int_0^\infty \frac{d\varphi}{dr} \left(\frac{d}{dr} + \frac{1}{r}\right) \left(R_0 \cdot R_2^{(E)\star}\right) dr. \end{aligned}\right\} (119)$$

The displacement of the ground level due to the potential $\mathscr{W}_n$ is thus in first approximation, if we use for R_0 the expression (109),

$$\left.\begin{aligned} \Delta E_0 &= -\frac{5 g_1 g_2 - f_1 f_2}{4\pi\hbar c}\cdot\left(\frac{M_m}{M}\right)^2\cdot Mc^2\cdot\frac{\alpha^4}{2}\int_0^\infty e^{-(\alpha+1)x}(x+1)\,dx \\ &= -\frac{5 g_1 g_2 - f_1 f_2}{4\pi\hbar c}\cdot\left(\frac{M_m}{M}\right)^2\cdot Mc^2\cdot\frac{\alpha^4(\alpha+2)}{2(\alpha+1)^2}. \end{aligned}\right\} \quad (120)$$

Assuming $\frac{M_m}{M} = \frac{1}{10}$ and $\alpha = 2.13$, we find, according to (108) and (54),

$$|\Delta E_0| \sim 0.01\cdot\frac{|5 g_1 \pm f_1|}{\sqrt{4\pi\hbar c}}\cdot Mc^2, \quad (121)$$

the double sign corresponding to the two possible choices of the sign of $f_2 : g_2$. We thus see that if, for instance, the factor $\frac{|5 g_1 \pm f_1|}{\sqrt{4\pi\hbar c}}$ is of the same order of magnitude as $\frac{|g_1|}{\sqrt{4\pi\hbar c}}$, i. e., according to (106), ~ 0.16, the displacement $|\Delta E_0|$ is quite considerable, being in fact more than $\frac{1}{2}$ of the whole binding energy $|E_0|$.

This circumstance would make a more rigorous treatment appear desirable, but one should not forget that the existence of a universal length r_0 might introduce just in the determination of ΔE_0—in contrast to effects depending on the static potential only—a quite appreciable modification. Although it is difficult to estimate the nature of such a modification, one might presume that one could get an idea of it simply by extending in (120) the integration over x only from κr_0 to infinity. According to (97) and (108), this would reduce the value of ΔE_0 by about a factor 2. It may be observed that a similar modification would leave the mean value of the static potential practically unchanged; with reference to the formulae (129), (130) below for the quadrupole moment of the deuteron, it will be seen that also this quantity is not appreciably affected by the modification just discussed.

At any rate, it is easily seen that even such a large correction as (121) to the binding energy E_0 would not essentially modify the numerical values of the constants $|g_1|$ and $|g_2|$, and would therefore not impose any essential limitation on the choice of the constant f_1.

The perturbed eigenfunction $\Phi_{E_0,\, l=0;\, j=1,\, m}$ of the ground state may be written

$$\left.\begin{aligned}\Phi_{E_0, l=0; j=1, m} = \Psi_{E_0, l=0; j=1, m} &+ \int_0^\infty \frac{(E, l=0\,|\,\mathcal{U}_n|\,0)}{E_0 - E}\,\Psi_{E, l=0; j=1, m}\, dE \\ &+ \int_0^\infty \frac{(E, l=2\,|\,\mathcal{U}_n|\,0)}{E_0 - E}\,\Psi_{E, l=2; j=1, m}\, dE,\end{aligned}\right\}\quad (122)$$

all states of types I a other than the ground state belonging, in our case, to the continuous spectrum. In the calculation of the electric quadrupole moment to the first order, only the last integral will give a contribution.

3. Electric quadrupole moment of the ground state.

The quantity defined as "electric quadrupole moment" of the normal state of a nucleus in interaction with an external electric field is [23])

$$\left.\begin{aligned} Q_{E_0;\, j} &= \int \Phi^{\star}_{E_0;\, j,\, m\, =\, j}\, (3\cos^2\theta - 1)\, r^2\, \Phi_{E_0;\, j,\, m\, =\, j}\, dv \\ &= \int \left| \Phi_{E_0;\, j,\, m\, =\, j} \right|^2 \frac{2\sqrt{2}}{\sqrt{5}}\, P_{2,0}(\cos\theta)\, r^2\, dv. \end{aligned}\right\} \tag{123}$$

According to (122), this gives for the ground state of the deuteron, in first approximation,

$$\left.\begin{aligned} Q = \int_0^{\infty} dE\, \frac{(E,\, l = 2\, |\, \mathcal{U}_n\, |\, 0)}{E_0 - E} \\ \int \Psi^{\star}_{E_0,\, l\, =\, 0;\, j\, =\, 1,\, m\, =\, 1}\, \Psi_{E,\, l\, =\, 2;\, j\, =\, 1,\, m\, =\, 1}\, \frac{2\sqrt{2}}{\sqrt{5}}\, P_{2,0}(\cos\theta)\, r^2\, dv + \text{conj.} \end{aligned}\right\} \tag{124}$$

Since the first order of magnitude in the velocities of the nuclear particles is about the same numerically as that of the parameter $\frac{G}{4\pi\hbar c}$, we should also take account of any quadrupole moment of second order in the velocities which could be present in the unperturbed system; it is immediately seen, however, from (110) to (113), that in our case such a quadrupole moment

$$\int \left| \Psi_{E_0;\, j,\, m\, =\, j} \right|^2 \frac{2\sqrt{2}}{\sqrt{5}}\, P_{2,0}(\cos\theta)\, r^2\, dv$$

reduces, to the order of magnitude indicated, to

$$\int \sum_{\sigma_3^P \sigma_3^N} \left| \overset{(0)}{z}_{\sigma_3^P \sigma_3^N} \right|^2 \cdot \frac{2\sqrt{2}}{\sqrt{5}}\, P_{2,0}(\cos\theta)\, r^2\, dv,$$

which vanishes because

$$\sum_{\sigma_3^P \sigma_3^N} \left| \overset{(0)}{z}_{\sigma_3^P \sigma_3^N} \right|^2$$

is independent of the angular variables.

Since, according to (113), (114) the product

$$\Psi^{\star}_{E_0, l=0; j=1, m=1} \cdot \Psi_{E, l=2; j=1, m=1}$$

reduces, after summation over the spin coordinates and integration over the angle φ, to

$$\frac{1}{2\sqrt{5}} P_{2,0}(\cos\theta) \cdot \frac{R_0^{\star} R_2^{(E)}}{r^2},$$

the expression (124) takes the simple form

$$Q = \frac{\sqrt{2}}{5} \int_0^{\infty} \frac{[(0\,|\,r^2\,|\,E, l=2)\,(E, l=2\,|\,\mathcal{H}_n\,|\,0) + \text{conj.}]\, dE}{E_0 - E}, \quad (125)$$

where

$$(0\,|\,r^2\,|\,E,\, l=2) = \int_0^{\infty} R_0^{\star}(r)\, R_2^{(E)}(r)\, r^2\, dr. \quad (126)$$

With the same notation, the integral occurring in the expression (119) of $(E,\, l = 2\,|\,\mathcal{H}_n\,|\,0)$ may be written

$$\left.\begin{aligned} \int_0^{\infty} \frac{d\varphi}{dr} \left(\frac{d}{dr} + \frac{1}{r}\right) \left(R_0 R_2^{(E)\star}\right) dr &= \int_0^{\infty} \left(-\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr}\right) R_0 R_2^{(E)\star}\, dr \\ &= -\left(E,\, l=2 \left|\, \varphi\left(\kappa^2 + \frac{3\kappa}{r} + \frac{3}{r^2}\right) \right| 0\right), \end{aligned}\right\} \quad (127)$$

if account is taken of the fact that $\left(\frac{d\varphi}{dr} R_0 R_2^{(E)\star}\right)_{r=0} = 0.$ Taking now the radial wave-functions real, we thus get

$$\left.\begin{aligned} Q &= \frac{8}{5} \cdot \frac{g_1 g_2 + f_1 f_2}{4\pi\hbar c} \cdot \left(\frac{M_m}{M}\right)^2 \cdot Mc^2 \cdot \\ &\cdot \frac{4\pi}{\kappa^3} \int_0^{\infty} \frac{(0\,|\,r^2\,|\,E,\, l=2)\left(E,\, l=2 \left|\, \varphi\left(\kappa^2 + \frac{3\kappa}{r} + \frac{3}{r^2}\right)\right| 0\right) dE}{|E_0| + E}. \end{aligned}\right\} \quad (128)$$

An estimation of the order of magnitude of Q may be obtained by writing, on account of the completeness of the system of eigenfunctions $R_2^{(E)}$,

$$Q = \frac{8}{5} \frac{g_1 g_2 + f_1 f_2}{4\pi \hbar c} \cdot \left(\frac{M_m}{M}\right)^2 \cdot \frac{Mc^2}{|E_0| + E_m} \cdot \frac{1}{\kappa^2} \left(0 \left| e^{-\kappa r} \left(\kappa r + 3 + \frac{3}{\kappa r}\right) \right| 0\right), \quad (129)$$

E_m being some eigenvalue, for which it is natural to assume the value corresponding to the maximum of the numerator of the integral in (128). Using (109) we find

$$\left. \begin{aligned} \left(0 \left| e^{-\kappa r} \left(\kappa r + 3 + \frac{3}{\kappa r}\right) \right| 0\right) &= \frac{\alpha^3}{2} \int_0^\infty e^{-(\alpha + 1)x} \left(x + 3 + \frac{3}{x}\right) x^2 \, dx \\ &= \frac{3}{2} \cdot \frac{\alpha^3 (\alpha^2 + 4\alpha + 5)}{(\alpha + 1)^4} \sim 2.7. \end{aligned} \right\} \quad (130)$$

The value of E_m was estimated by taking for $R_2^{(E)}$ the Bessel function $\sqrt{kr}\, J_{5/2}(kr) \left(\text{with } k = \frac{1}{\hbar} \sqrt{ME}\right)$ corresponding to a complete neglect of the static potential. It was found that the maximum of the numerator of the integral in (128), calculated in this way, lies at about $k_m \sim 1.3\,\kappa$, or

$$\left. \begin{aligned} E_m &\sim 1.7 \left(\frac{M_m}{M}\right)^2 Mc^2 \\ &\sim 0.017\, Mc^2. \end{aligned} \right\} \quad (131)$$

For the absolute value of Q, we therefore get

$$|Q| \sim 0.6 \left| \frac{g_1 \pm f_1}{\sqrt{4\pi \hbar c}} \right| \cdot \frac{1}{\kappa^2}, \quad (132)$$

or, if we assume $\left| \frac{g_1 \pm f_1}{\sqrt{4\pi \hbar c}} \right| \sim \left| \frac{g_1}{\sqrt{4\pi \hbar c}} \right| \sim 0.16,$

$$|Q| \backsim 0.1 \cdot \frac{1}{\kappa^2} \backsim 4 \cdot 10^{-27} cm^2. \qquad (133)$$

The only meaning of this rough calculation is to show that the value of Q, on the present theory, may well be of the order of magnitude indicated by the provisory empirical results[13), 10)]; the theory may of course be fitted to account for any sign of the quadrupole moment, practically without influence on its absolute value. We see at any rate that, while the existence of a quadrupole moment is of fundamental importance in pointing to a relatively large contribution of directional couplings to the interaction between nuclear particles, the incorporation of such an effect in the meson theory does not involve any considerable restriction in the choice of the formalism.

We should like to express our deep gratitude to Professor Niels Bohr not only for his kind interest in this work but also for the constant inspiration derived from the many discussions and conversations at the Institute. We extend our best thanks to Dr. L. Hulthén, who kindly checked our calculations on the deuteron. One of us desires also to thank the Belgian American Educational Foundation for a grant which enabled him to visit several American Universities and there in particular enjoy valuable discussions on the subject of the present paper.

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Contents.

Indleveret til Selskabet den 1. December 1939.
Færdig fra Trykkeriet den 30. Marts 1940.